A First Course in Fourier Analysis

This unique book provides a meaningful resource for applied mathematics through Fourier analysis. It develops a unified theory of discrete and continuous (univariate) Fourier analysis, the fast Fourier transform, and a powerful elementary theory of generalized functions, including the use of weak limits. It then shows how these mathematical ideas can be used to expedite the study of sampling theory, PDEs, wavelets, probability, diffraction, etc. Unique features include a unified development of Fourier synthesis/analysis for functions on $\mathbb{R}$, $\mathbb{T}_p$, $\mathbb{Z}$, and $\mathbb{P}_N$; an unusually complete development of the Fourier transform calculus (for finding Fourier transforms, Fourier series, and DFTs); memorable derivations of the FFT; a balanced treatment of generalized functions that fosters mathematical understanding as well as practical working skills; a careful introduction to Shannon's sampling theorem and modern variations; a study of the wave equation, diffusion equation, and diffraction equation by using the Fourier transform calculus, generalized functions, and weak limits; an exceptionally efficient development of Daubechies' compactly supported orthogonal wavelets; generalized probability density functions with corresponding versions of Bochner's theorem and the central limit theorem; and a real-world application of Fourier analysis to the study of musical tones. A valuable reference on Fourier analysis for a variety of scientific professionals, including Mathematicians, Physicists, Chemists, Geologists, Electrical Engineers, Mechanical Engineers, and others.

David Kammler is a Professor and Distinguished Teacher in the Mathematics Department at Southern Illinois University.

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David W. Kammler

Department of Mathematics
Southern Illinois University at Carbondale

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Mathematics: Source and Substance

Profound study of nature is the most fertile source of mathematical discoveries.

Joseph Fourier, *The Analytical Study of Heat*, p. 7

Mathematics is the science of patterns. The mathematician seeks patterns in number, in space, in science, in computers, and in imagination. Mathematical theories explain the relations among patterns; functions and maps, operators and morphisms bind one type of pattern to another to yield lasting mathematical structures. Applications of mathematics use these patterns to *explain* and predict natural phenomena that fit the patterns. Patterns suggest other patterns, often yielding patterns of patterns. In this way mathematics follows its own logic, beginning with patterns from science and completing the portrait by adding all patterns that derive from initial ones.

Lynn A. Steen, The science of patterns, *Science* **240**(1988), 616.

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Selected Applications

Preface

To the Student

This book is about one big idea: You can synthesize a variety of complicated functions from pure sinusoids in much the same way that you produce a major chord by striking nearby C, E, G keys on a piano. A geometric version of this idea forms the basis for the ancient Hipparchus-Ptolemy model of planetary motion (*Almagest*, 2nd century; see Fig. 1.2). It was Joseph Fourier (*Analytical Theory of Heat*, 1815), however, who developed modern methods for using trigonometric series and integrals as he studied the flow of heat in solids. Today, Fourier analysis is a highly evolved branch of mathematics with an incomparable range of applications and with an impact that is second to none (see Appendix 1). If you are a student in one of the mathematical, physical, or engineering sciences, you will almost certainly find it necessary to learn the elements of this subject. My goal in writing this book is to help you acquire a working knowledge of Fourier analysis early in your career.

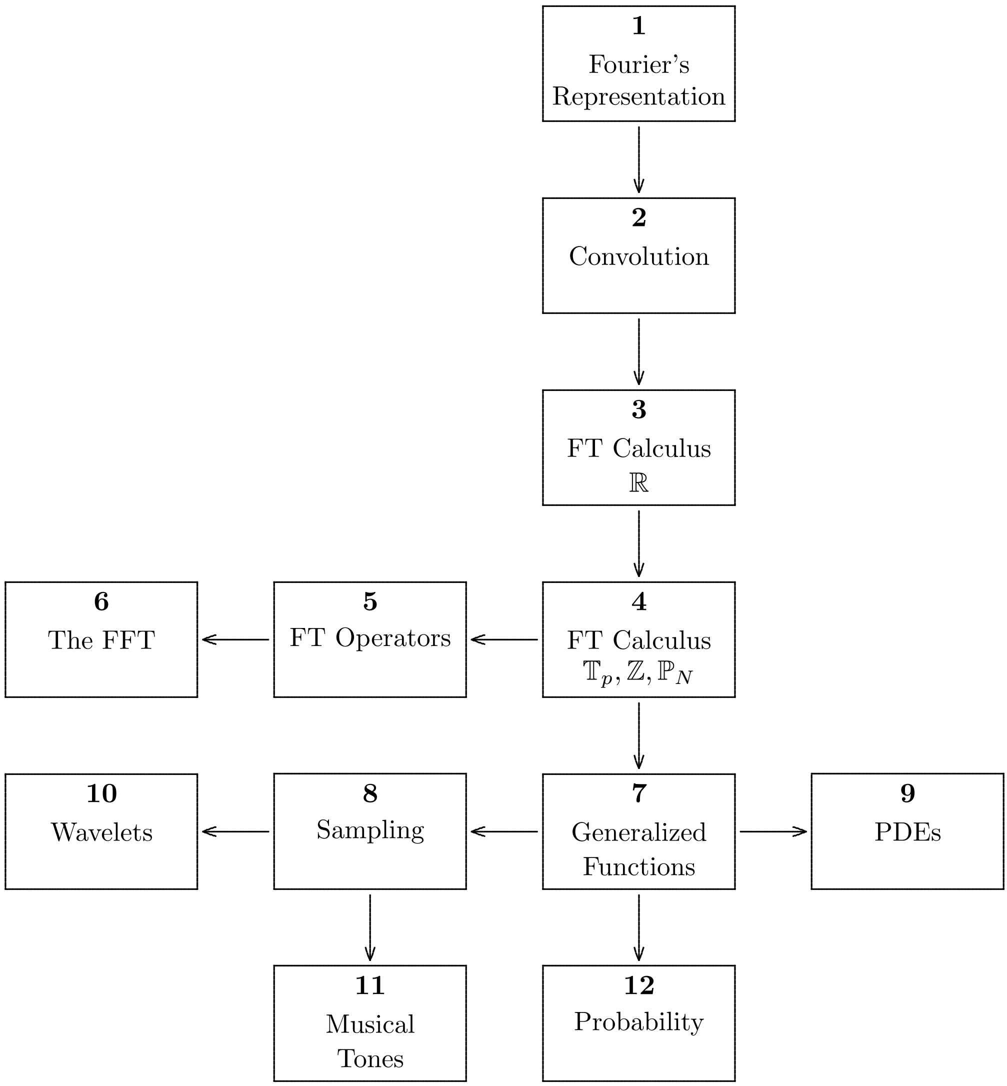
If you have mastered the usual core courses in calculus and linear algebra, you have the maturity to follow the presentation without undue difficulty. A few of the proofs and more theoretical exercises require concepts (uniform continuity, uniform convergence, . . .) from an analysis or advanced calculus course. You may choose to skip over the difficult steps in such arguments and simply accept the stated results. The text has been designed so that you can do this without severely impacting your ability to learn the important ideas in the subsequent chapters. In addition, I will use a *potpourri* of notions from undergraduate courses in differential equations [solve $y'(x) + \alpha y(x) = 0$, $y'(x) = xy(x)$, $y''(x) + \alpha^2 y(x) = 0, \ldots$], complex analysis (Euler's formula: $e^{i\theta} = \cos\theta + i \sin\theta$, arithmetic for complex numbers, . . .), number theory (integer addition and multiplication modulo N, Euclid's gcd algorithm, . . .), probability (random variable, mean, variance, . . .), physics ($F = ma$, conservation of energy, Huygens' principle, . . .), signals and systems (LTI systems, low-pass filters, the Nyquist rate, . . .), etc. You will have no trouble picking up these concepts as they are introduced in the text and exercises.

If you wish, you can find additional information about almost any topic in this book by consulting the annotated references at the end of the corresponding chapter. You will often discover that I have abandoned a traditional presentation in favor of one that is in keeping with my goal of making these ideas accessible to undergraduates. For example, the usual presentation of the Schwartz theory of distributions assumes some familiarity with the Lebesgue integral and with a graduate-level functional analysis course. In contrast, my development of δ, Ш, ... in Chapter 7 uses only notions from elementary calculus. Once you master this theory, you can use generalized functions to study sampling, PDEs, wavelets, probability, diffraction, ...

The exercises (541 of them) are my greatest gift to you! Read each chapter carefully to acquire the basic concepts, and then solve as many problems as you can. You may find it beneficial to organize an interdisciplinary study group, e.g., mathematician + physicist + electrical engineer. Some of the exercises provide routine drill: You must learn to find convolution products, to use the FT calculus, to do routine computations with generalized functions, etc. Some supply historical perspective: You can play Gauss and discover the FFT, analyze Michelson and Stratton's analog supercomputer for summing Fourier series, etc. Some ask for mathematical details: Give a sufficient condition for ... , given an example of ... , show that, ... Some involve your personal harmonic analyzers: Experimentally determine the bandwidth of your eye, describe what would you hear if you replace notes with frequencies $\mathrm{F}_1, \mathrm{F}_2, \ldots$ by notes with frequencies C/F_1, C/F_2, ... Some prepare you for computer projects: Compute π to 1000 digits, prepare a movie for a vibrating string, generate the sound file for Risset's endless glissando, etc. Some will set you up to discover a pattern, formulate a conjecture, and prove a theorem. (It's quite a thrill when you get the hang of it!) I expect you to spend a lot of time working exercises, but I want to help you work efficiently. Complicated results are broken into simple steps so you can do (a), then (b), then (c), ... until you reach the goal. I frequently supply hints that will lead you to a productive line of inquiry. You will sharpen your problem-solving skills as you take this course.

Synopsis

The chapters of the book are arranged as follows:



The mathematical core is given in Chapters 1–7 and selected applications are developed in Chapters 8–12.

We present the basic themes of Fourier analysis in the first two chapters. Chapter 1 opens with Fourier's synthesis and analysis equations for functions on the real line $\mathbb{R}$, on the circle $\mathbb{T}_p$, on the integers $\mathbb{Z}$, and on the polygon $\mathbb{P}_N$. We discretize by sampling (obtaining functions on $\mathbb{Z}, \mathbb{P}_N$ from functions on $\mathbb{R}, \mathbb{T}_p$), we periodize by summing translates (obtaining functions on $\mathbb{T}_p, \mathbb{P}_N$ from functions on $\mathbb{R}, \mathbb{Z}$), and we informally derive the corresponding Poisson identities. We combine these mappings to form the Fourier–Poisson cube, a structure that links the Fourier transforms, Fourier series, and discrete Fourier transforms students encounter in their undergraduate classes. We prove that these equations are valid when certain elementary sufficient conditions are satisfied. We complete the presentation of basic themes by describing the convolution product of functions on $\mathbb{R}, \mathbb{T}_p, \mathbb{Z}$, and $\mathbb{P}_N$ in Chapter 2.

Chapters 3 and 4 are devoted to the development of computational skills. We introduce the Fourier transform calculus for functions on $\mathbb{R}$ by finding transforms of the box, $\Pi(x)$, the truncated exponential, $e^{-x}\,h(x)$, and the unit gaussian $e^{-\pi x^2}$. We present the rules (linearity, translation, dilation, convolution, inversion, ...) and use them to obtain transforms for a large class of functions on $\mathbb{R}$. Various methods are used to find Fourier series. In addition to direct integration (with Kronecker's rule), we present (and emphasize) Poisson's formula, Eagle's method, and the use of elementary Laurent series (from calculus). Corresponding rules facilitate the manipulation of the Fourier representations for functions on $\mathbb{T}_p$ and $\mathbb{Z}$. An understanding of the Fourier transform calculus for functions on $\mathbb{P}_N$ is essential for anyone who wishes to use the FFT. We establish a few well-known DFT pairs and develop the corresponding rules. We illustrate the power of this calculus by deriving the Euler–Maclaurin sum formula from elementary numerical analysis and evaluating the Gauss sums from elementary number theory.

In Chapter 5 we use operators, i.e., function-to-function mappings, to organize the multiplicity of specialized Fourier transform rules. We characterize the basic symmetries of Fourier analysis and develop a deeper understanding of the Fourier transform calculus. We also use the operator notation to facilitate a study of Sine, Cosine, Hartley, and Hilbert transforms.

The subject of Chapter 6 is the FFT (which Gilbert Strang calls the most important algorithm of the 20th century!). After describing the $O(N^2)$ scheme of Horner, we use the DFT calculus to produce an N-point DFT with only $O(N \log_2 N)$ operations. We use an elementary zipper identity to obtain a sparse factorization of the DFT matrix and develop a corresponding algorithm (including the clever enhancements of Bracewell and Buneman) for fast machine computation. We briefly introduce some of the more specialized DFT factorizations that can be obtained by using Kronecker products.

An elementary exposition of generalized functions (the tempered distributions of Schwartz) is given in Chapter 7, the heart of the book. We introduce the Dirac δ [as the second derivative of the ramp $r(x) := \max(x, 0)$], the comb Ш; the reciprocal "$1/x$", the Fresnel function $e^{i\pi x^2}$, ... and carefully extend the FT calculus rules to this new setting. We introduce generalized (weak) limits so that we can work with infinite series, infinite products, ordinary derivatives, partial derivatives, ...

Selected applications of Fourier analysis are given in the remaining chapters. (You can find whole textbooks devoted to each of these topics.) Mathematical models based on Fourier synthesis, analysis done with generalized functions, and FFT computations are used to foster insight and understanding. You will experience the enormous "leverage" Fourier analysis can give as you study this material!

Sampling theory, the mathematical basis for digital signal processing, is the focus of Chapter 8. We present weak and strong versions of Shannon's theorem together with the clever generalization of Papoulis. Using these ideas (and characteristics of the human ear) we develop the elements of computer music in Chapter 11. We use additive synthesis and Chowning's FM synthesis to generate samples for musical tones, and we use spectrograms to visualize the structure of the corresponding sound files.

Fourier analysis was invented to solve PDEs, the subject of Chapter 9. We formulate mathematical models for the motion of a vibrating string, for the diffusion of heat (Fourier's work), and for Fresnel diffraction. (The Schrödinger equation from quantum mechanics seems much less intimidating when interpreted within the context of elementary optics!) With minimal effort, we solve these PDEs, establish suitable conservation laws, and examine representative solutions. (The cover illustration was produced by using the FFT to generate slices for the diffraction pattern that results when two gaussian laser beams interfere.)

Chapter 10 is devoted to the study of wavelets, a relatively new branch of mathematics. We introduce the basic ideas using the piecewise constant functions associated with the Haar wavelets. We then use the theory of generalized functions to develop the compactly supported orthogonal wavelets created by I. Daubechies in 1988. Fourier analysis plays an essential role in the study of corresponding filter banks that are used to process audio and image files.

We present the elements of probability theory in Chapter 12 using generalized densities, e.g., $f(x) := (1/2)[\delta(x+1) + \delta(x-1)]$ serves as the probability density for a coin toss. We use Fourier analysis to find moments, convolution products, characteristic functions, and to establish the uncertainty relation (for suitably regular probability densities on $\mathbb{R}$). We then use the theory of generalized functions to prove the central limit theorem, the foundation for modern statistics!

To the Instructor

This book is the result of my efforts to create a modern elementary introduction to Fourier analysis for students from mathematics, science, and engineering. There is more than enough material for a tight one-semester survey or for a leisurely two-semester course that allocates more time to the applications. You can adjust the level and the emphasis of the course to your students by the topics you cover and by your assignment of homework exercises. You can use Chapters 1–4, 7, and 9 to update a lackluster boundary value problems course. You can use Chapters 1–4, 7, 8, and 10 to give a serious introduction to sampling theory and wavelets. You can use selected portions of Chapters 2–4, 6, 8, and 11 (with composition exercises!) for a fascinating elementary introduction to the mathematics of computer-generated music. You can use the book for an undergraduate capstone course that emphasizes group learning of the interdisciplinary topics and mastering of some of the more difficult exercises. Finally, you can use Chapters 7–12 to give a graduate-level introduction to generalized functions for scientists and engineers.

This book is not a traditional mathematics text. You will find a minimal amount of jargon and note the absence of a logically complete theorem-proof presentation of elementary harmonic analysis. Basic computational skills are developed for solving real problems, not just for drill. There is a strong emphasis on the visualization of equations, mappings, theorems, ... and on the interpretation of mathematical ideas within the context of some application. In general, the presentation is informal, but there are careful proofs for theorems that have strategic importance, and there are a number of exercises that lead students to develop the implications of ideas introduced in the text.

Be sure to cover one or more of the applications chapters. Students enjoy learning about the essential role Fourier analysis plays in modern mathematics, science, and engineering. You will find that it is much easier to develop and to maintain the market for a course that emphasizes these applications.

When I teach this material I devote 24 lectures to the mathematical core (deleting portions of Chapters 1, 5, and 6) and 18 lectures to the applications (deleting portions of Chapters 10, 11, and 12). I also spend 3–4 hours per week conducting informal problem sessions, giving individualized instruction, etc. I lecture from transparencies and use a PC (with *FOURIER*) for visualization and sonification. This is helpful for the material in Chapters 2, 5, 6, and 12 and essential for the material in Chapters 9, 10, and 11. I use a laser with apertures on 35 mm slides to show a variety of diffraction patterns when I introduce the topic of diffraction in Chapter 9. This course is a great place to demonstrate the synergistic roles of experimentation, mathematical modeling, and computer simulation in modern science and engineering.

I have one word of caution. As you teach this material you will face the constant temptation to prove too much too soon. My informal use of $\stackrel{?}{=}$ cries out for the precise statement and proof of some relevant sufficient condition. (In most cases there is a corresponding exercise, with hints, for the student who would really like to see the details.) For every hour that you spend presenting 19th-century advanced calculus arguments, however, you will have one less hour for explaining the 20th-century mathematics of generalized functions, sampling theory, wavelets, ... You must decide which of these alternatives will best serve your students.

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I wish to thank Southern Illinois University at Carbondale for providing a nurturing environment during the evolution of ideas that led to this book. Sabbatical leaves and teaching fellowships were essential during the early phases of the work. National Science Foundation funding (NSF-USE 89503, NSF-USE 9054179) for four faculty short courses at the Touch of Nature center at SIUC during 1989–1992, and NECUSE funding for a faculty short course at Bates College in 1995 enabled me to share preliminary versions of these ideas with faculty peers. I have profited enormously from their thoughtful comments and suggestions, particularly those of Davis Cope, Bo Green, Carruth McGehee, Dale Mugler, Mark Pinsky, David Snider, Patrick Sullivan, Henry Warchall, and Jo Ward. I deeply appreciate the National Science Foundation course development grants (NSF-USE 9156064, NSF-CCLI 127048) that provided support for the creation of many of the exercise sets as well as for equipment and programming services that were used to develop the second half of the book.

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I hope that you enjoy this approach for learning Fourier analysis. If you have corrections, ideas for new exercises, suggestions for improving the presentation, etc., I would love to hear from you!

David W. Kammler
Mathematics Department
Southern Illinois University at Carbondale
Carbondale, Illinois 62901–4408
dkammler@math.siu.edu