

Cambridge University Press

978-0-521-88336-8 - Harmonic Analysis on Finite Groups: Representation Theory,  
Gelfand Pairs and Markov Chains

Tullio Ceccherini-Silberstein, Fabio Scarabotti and Filippo Tolli

Frontmatter

[More information](#)

---

## Harmonic Analysis on Finite Groups

CAMBRIDGE STUDIES IN ADVANCED MATHEMATICS 108

### *EDITORIAL BOARD*

B. BOLLOBÁS, W. FULTON, A. KATOK, F. KIRWAN, P. SARNAK,  
B. SIMON, B. TOTARO

### HARMONIC ANALYSIS ON FINITE GROUPS

Line up a deck of 52 cards on a table. Randomly choose two cards and switch them. How many switches are needed in order to mix up the deck?

Starting from a few concrete problems, such as the random walk on the discrete circle and the Ehrenfest diffusion model, this book develops the necessary tools for the asymptotic analysis of these processes. This detailed study culminates with the case-by-case analysis of the cut-off phenomenon discovered by Persi Diaconis.

This self-contained text is ideal for students and researchers working in the areas of representation theory, group theory, harmonic analysis and Markov chains. Its topics range from the basic theory needed for students new to this area, to advanced topics such as Gelfand pairs, harmonics on posets and  $q$ -analogs, the complete analysis of the random matchings, and the representation theory of the symmetric group.

Cambridge University Press

978-0-521-88336-8 - Harmonic Analysis on Finite Groups: Representation Theory,  
Gelfand Pairs and Markov Chains

Tullio Ceccherini-Silberstein, Fabio Scarabotti and Filippo Tolli

Frontmatter

[More information](#)

## CAMBRIDGE STUDIES IN ADVANCED MATHEMATICS

*Editorial Board:*

B. Bollobas, W. Fulton, A. Katok, F. Kirwan, P. Sarnak, B. Simon, B. Totaro

All the titles listed below can be obtained from good booksellers or from Cambridge University Press. For a complete series listing visit: <http://www.cambridge.org/series/sSeries.asp?code=CSAM>

*Already published*

- 60 M.P. Brodmann & R.Y. Sharp *Local cohomology*
- 61 J.D. Dixon *et al.* *Analytic pro-p groups*
- 62 R. Stanley *Enumerative combinatorics II*
- 63 R.M. Dudley *Uniform central limit theorems*
- 64 J. Jost & X. Li-Jost *Calculus of variations*
- 65 A.J. Berrick & M.E. Keating *An introduction to rings and modules*
- 66 S. Morosawa *Holomorphic dynamics*
- 67 A.J. Berrick & M.E. Keating *Categories and modules with K-theory in view*
- 68 K. Sato *Levy processes and infinitely divisible distributions*
- 69 H. Hida *Modular forms and Galois cohomology*
- 70 R. Iorio & V. Iorio *Fourier analysis and partial differential equations*
- 71 R. Blei *Analysis in integer and fractional dimensions*
- 72 F. Borceaux & G. Janelidze *Galois theories*
- 73 B. Bollobás *Random graphs*
- 74 R.M. Dudley *Real analysis and probability*
- 75 T. Sheil-Small *Complex polynomials*
- 76 C. Voisin *Hodge theory and complex algebraic geometry, I*
- 77 C. Voisin *Hodge theory and complex algebraic geometry, II*
- 78 V. Paulsen *Completely bounded maps and operator algebras*
- 79 F. Gesztesy & H. Holden *Soliton Equations and Their Algebraic-Geometric Solutions, I*
- 81 S. Mukai *An Introduction to Invariants and Moduli*
- 82 G. Tourlakis *Lectures in Logic and Set Theory, I*
- 83 G. Tourlakis *Lectures in Logic and Set Theory, II*
- 84 R.A. Bailey *Association Schemes*
- 85 J. Carlson, S. Müller-Stach & C. Peters *Period Mappings and Period Domains*
- 86 J.J. Duistermaat & J.A.C. Kolk *Multidimensional Real Analysis I*
- 87 J.J. Duistermaat & J.A.C. Kolk *Multidimensional Real Analysis II*
- 89 M. Golumbic & A. Trenk *Tolerance Graphs*
- 90 L. Harper *Global Methods for Combinatorial Isoperimetric Problems*
- 91 I. Moerdijk & J. Mrcun *Introduction to Foliations and Lie Groupoids*
- 92 J. Kollar, K.E. Smith & A. Corti *Rational and Nearly Rational Varieties*
- 93 D. Applebaum *Levy Processes and Stochastic Calculus*
- 94 B. Conrad *Modular Forms and the Ramanujan Conjecture*
- 95 M. Schechter *An Introduction to Nonlinear Analysis*
- 96 R. Carter *Lie Algebras of Finite and Affine Type*
- 97 H.L. Montgomery, R.C. Vaughan & M. Schechter *Multiplicative Number Theory I*
- 98 I. Chavel *Riemannian Geometry*
- 99 D. Goldfeld *Automorphic Forms and L-Functions for the Group  $GL(n, R)$*
- 100 M. Marcus & J. Rosen *Markov Processes, Gaussian Processes, and Local Times*
- 101 P. Gille & T. Szamuely *Central Simple Algebras and Galois Cohomology*
- 102 J. Bertoin *Random Fragmentation and Coagulation Processes*
- 103 E. Frenkel *Langlands Correspondence for Loop Groups*
- 104 A. Ambrosetti & A. Malchiodi *Nonlinear Analysis and Semilinear Elliptic Problems*
- 105 T. Tao & V.H. Vu *Additive Combinatorics*
- 106 E.B. Davies *Linear Operators and their Spectra*
- 107 K. Kodaira *Complex Analysis*

Cambridge University Press

978-0-521-88336-8 - Harmonic Analysis on Finite Groups: Representation Theory,  
Gelfand Pairs and Markov Chains

Tullio Ceccherini-Silberstein, Fabio Scarabotti and Filippo Togli

Frontmatter

[More information](#)

# Harmonic Analysis on Finite Groups

## Representation Theory, Gelfand Pairs and Markov Chains

TULLIO CECCHERINI-SILBERSTEIN

*Università del Sannio, Benevento*

FABIO SCARABOTTI

*Università di Roma "La Sapienza", Rome*

FILIPPO TOLLI

*Università Roma Tre, Rome*



**CAMBRIDGE**  
UNIVERSITY PRESS

Cambridge University Press

978-0-521-88336-8 - Harmonic Analysis on Finite Groups: Representation Theory,  
Gelfand Pairs and Markov Chains

Tullio Ceccherini-Silberstein, Fabio Scarabotti and Filippo Tolli

Frontmatter

[More information](#)

CAMBRIDGE UNIVERSITY PRESS

Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press

The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

[www.cambridge.org](http://www.cambridge.org)

Information on this title: [www.cambridge.org/9780521883368](http://www.cambridge.org/9780521883368)

© T. Ceccherini-Silberstein, F. Scarabotti and F. Tolli 2008

This publication is in copyright. Subject to statutory exception  
and to the provisions of relevant collective licensing agreements,  
no reproduction of any part may take place without  
the written permission of Cambridge University Press.

First published 2008

Printed in the United Kingdom at the University Press, Cambridge

*A catalogue record for this publication is available from the British Library*

ISBN 978-0-521-88336-8 hardback

Cambridge University Press has no responsibility for the persistence or  
accuracy of URLs for external or third-party internet websites referred to  
in this publication, and does not guarantee that any content on such  
websites is, or will remain, accurate or appropriate.

Cambridge University Press

978-0-521-88336-8 - Harmonic Analysis on Finite Groups: Representation Theory,  
Gelfand Pairs and Markov Chains

Tullio Ceccherini-Silberstein, Fabio Scarabotti and Filippo Tolli

Frontmatter

[More information](#)

---

To Katuscia, Giacomo, and Tommaso

To my parents and Cristina

To my Mom, Rossella, and Stefania

Cambridge University Press

978-0-521-88336-8 - Harmonic Analysis on Finite Groups: Representation Theory,  
Gelfand Pairs and Markov Chains

Tullio Ceccherini-Silberstein, Fabio Scarabotti and Filippo Tolli

Frontmatter

[More information](#)

# Contents

	<i>Preface</i>	<i>page xi</i>
	<b>Part I Preliminaries, examples and motivations</b>	<b>1</b>
<b>1</b>	<b>Finite Markov chains</b>	<b>3</b>
	1.1 Preliminaries and notation	3
	1.2 Four basic examples	4
	1.3 Markov chains	10
	1.4 Convergence to equilibrium	18
	1.5 Reversible Markov chains	21
	1.6 Graphs	26
	1.7 Weighted graphs	28
	1.8 Simple random walks	33
	1.9 Basic probabilistic inequalities	37
	1.10 Lumpable Markov chains	41
<b>2</b>	<b>Two basic examples on abelian groups</b>	<b>46</b>
	2.1 Harmonic analysis on finite cyclic groups	46
	2.2 Time to reach stationarity for the simple random walk on the discrete circle	55
	2.3 Harmonic analysis on the hypercube	58
	2.4 Time to reach stationarity in the Ehrenfest diffusion model	61
	2.5 The cutoff phenomenon	66
	2.6 Radial harmonic analysis on the circle and the hypercube	69

Cambridge University Press

978-0-521-88336-8 - Harmonic Analysis on Finite Groups: Representation Theory,  
Gelfand Pairs and Markov Chains

Tullio Ceccherini-Silberstein, Fabio Scarabotti and Filippo Tolli

Frontmatter

[More information](#)

viii

*Contents*

	<b>Part II Representation theory and Gelfand pairs</b>	75
<b>3</b>	<b>Basic representation theory of finite groups</b>	77
	3.1 Group actions	77
	3.2 Representations, irreducibility and equivalence	83
	3.3 Unitary representations	83
	3.4 Examples	87
	3.5 Intertwiners and Schur's lemma	88
	3.6 Matrix coefficients and their orthogonality relations	89
	3.7 Characters	92
	3.8 More examples	96
	3.9 Convolution and the Fourier transform	98
	3.10 Fourier analysis of random walks on finite groups	103
	3.11 Permutation characters and Burnside's lemma	105
	3.12 An application: the enumeration of finite graphs	107
	3.13 Wielandt's lemma	110
	3.14 Examples and applications to the symmetric group	113
<b>4</b>	<b>Finite Gelfand pairs</b>	117
	4.1 The algebra of bi- $K$ -invariant functions	118
	4.2 Intertwining operators for permutation representations	120
	4.3 Finite Gelfand pairs: definition and examples	123
	4.4 A characterization of Gelfand pairs	125
	4.5 Spherical functions	127
	4.6 The canonical decomposition of $L(X)$ via spherical functions	132
	4.7 The spherical Fourier transform	135
	4.8 Garsia's theorems	140
	4.9 Fourier analysis of an invariant random walk on $X$	143
<b>5</b>	<b>Distance regular graphs and the Hamming scheme</b>	147
	5.1 Harmonic analysis on distance-regular graphs	147
	5.2 The discrete circle	161
	5.3 The Hamming scheme	162
	5.4 The group-theoretical approach to the Hamming scheme	165
<b>6</b>	<b>The Johnson scheme and the Bernoulli–Laplace diffusion model</b>	168
	6.1 The Johnson scheme	168
	6.2 The Gelfand pair $(S_n, S_{n-m} \times S_m)$ and the associated intertwining functions	176

Cambridge University Press

978-0-521-88336-8 - Harmonic Analysis on Finite Groups: Representation Theory,  
Gelfand Pairs and Markov Chains

Tullio Ceccherini-Silberstein, Fabio Scarabotti and Filippo Tolli

Frontmatter

[More information](#)*Contents*

ix

6.3	Time to reach stationarity for the Bernoulli–Laplace diffusion model	180
6.4	Statistical applications	184
6.5	The use of Radon transforms	187
<b>7</b>	<b>The ultrametric space</b>	191
7.1	The rooted tree $\mathbb{T}_{q,n}$	191
7.2	The group $Aut(\mathbb{T}_{q,n})$ of automorphisms	192
7.3	The ultrametric space	194
7.4	The decomposition of the space $L(\Sigma^n)$ and the spherical functions	196
7.5	Recurrence in finite graphs	199
7.6	A Markov chain on $\Sigma^n$	202
	<b>Part III Advanced theory</b>	207
<b>8</b>	<b>Posets and the <math>q</math>-analogs</b>	209
8.1	Generalities on posets	209
8.2	Spherical posets and regular semi-lattices	216
8.3	Spherical representations and spherical functions	222
8.4	Spherical functions via Moebius inversion	229
8.5	$q$ -binomial coefficients and the subspaces of a finite vector space	239
8.6	The $q$ -Johnson scheme	243
8.7	A $q$ -analog of the Hamming scheme	251
8.8	The nonbinary Johnson scheme	257
<b>9</b>	<b>Complements of representation theory</b>	267
9.1	Tensor products	267
9.2	Representations of abelian groups and Pontrjagin duality	274
9.3	The commutant	276
9.4	Permutation representations	279
9.5	The group algebra revisited	283
9.6	An example of a not weakly symmetric Gelfand pair	291
9.7	Real, complex and quaternionic representations: the theorem of Frobenius and Schur	294
9.8	Greenhalgebras	304
9.9	Fourier transform of a measure	314
<b>10</b>	<b>Basic representation theory of the symmetric group</b>	319
10.1	Preliminaries on the symmetric group	319



Cambridge University Press

978-0-521-88336-8 - Harmonic Analysis on Finite Groups: Representation Theory,  
Gelfand Pairs and Markov Chains

Tullio Ceccherini-Silberstein, Fabio Scarabotti and Filippo Tolli

Frontmatter

[More information](#)

x

*Contents*

10.2	Partitions and Young diagrams	322
10.3	Young tableaux and the Specht modules	325
10.4	Representations corresponding to transposed tableaux	333
10.5	Standard tableaux	335
10.6	Computation of a Fourier transform on the symmetric group	343
10.7	Random transpositions	348
10.8	Differential posets	358
10.9	James' intersection kernels theorem	365
<b>11</b>	<b>The Gelfand pair <math>(S_{2n}, S_2 \wr S_n)</math> and random matchings</b>	<b>371</b>
11.1	The Gelfand pair $(S_{2n}, S_2 \wr S_n)$	371
11.2	The decomposition of $L(X)$ into irreducible components	373
11.3	Computing the spherical functions	374
11.4	The first nontrivial value of a spherical function	375
11.5	The first nontrivial spherical function	376
11.6	Phylogenetic trees and matchings	378
11.7	Random matchings	380
	<i>Appendix 1 The discrete trigonometric transforms</i>	392
	<i>Appendix 2 Solutions of the exercises</i>	407
	<i>Bibliography</i>	421
	<i>Index</i>	433

Cambridge University Press

978-0-521-88336-8 - Harmonic Analysis on Finite Groups: Representation Theory,  
Gelfand Pairs and Markov Chains

Tullio Ceccherini-Silberstein, Fabio Scarabotti and Filippo Tolli

Frontmatter

[More information](#)

## Preface

In September 2003 we started writing a research expository paper on “Finite Gelfand pairs and their applications to probability and statistics” [43] for the proceedings of a conference held in Batumi (Georgia). After a preliminary version of that paper had been circulated, we received several emails of appreciation and encouragement from experts in the field. In particular, Persi Diaconis suggested that we expand that paper to a monograph on Gelfand pairs. In his famous 1988 monograph “Group representations in probability and statistics” [55] there is a short treatment of the theory of Gelfand pairs but, to his and our knowledge, no book entirely dedicated to Gelfand pairs was ever written. We thus started to expand the paper, including some background material to make the book self-contained, and adding some topics closely related to the kernel of the monograph. As the “close relation” is in some sense inductive, we pushed our treatment much further than what Persi was probably expecting. In all cases, we believe that our monograph is in some sense unique as it assembles, for the first time, the various topics that appear in it.

The book that came out is a course in finite harmonic analysis. It is completely self-contained (it only requires very basic rudiments of group theory and of linear algebra). There is also a large number of exercises (with solutions or generous hints) which constitute complements and/or further developments of the topics treated.

For this reason it can be used for a course addressed to both advanced undergraduates and to graduate students in pure mathematics as well as in probability and statistics. On the other hand, due to its completeness, it can also serve as a reference for mature researchers.

Cambridge University Press

978-0-521-88336-8 - Harmonic Analysis on Finite Groups: Representation Theory,  
Gelfand Pairs and Markov Chains

Tullio Ceccherini-Silberstein, Fabio Scarabotti and Filippo Tolli

Frontmatter

[More information](#)

It presents a very general treatment of the theory of finite Gelfand pairs and their applications to Markov chains with emphasis on the cutoff phenomenon discovered by Persi Diaconis.

The book by Audrey Terras [220], which bears a similar title, is in some sense orthogonal to our monograph, both in style and contents. For instance, we do not treat applications to number theory, while Terras does not treat the representation theory of the symmetric group.

We present six basic examples of diffusion processes, namely the random walk on the circle, the Ehrenfest and the Bernoulli–Laplace models of diffusion, a Markov chain on the ultrametric space, random transpositions and random matchings. Each of these examples bears its own peculiarity and needs specific tools of an algebraic/harmonic-analytic/probabilistic nature in order to analyze the asymptotic behavior of the corresponding process.

These tools, which we therefore develop in a very self-contained presentation are: spectral graph theory and reversible Markov chains, Fourier analysis on finite abelian groups, representation theory and Fourier analysis of finite groups, finite Gelfand pairs and their spherical functions, and representation theory of the symmetric group. We also present a detailed account of the (distance-regular) graph theoretic approach to spherical functions and on the use of finite posets.

All this said, one can use this monograph as a textbook for at least three different courses on:

- (i) **Finite Markov chains** (an elementary introduction oriented to the cutoff phenomenon): Chapters 1 and 2, parts of Chapters 5 and 6, and Appendix 1 (the discrete trigonometric transforms).
- (ii) **Finite Gelfand pairs** (and applications to probability): Chapters 1–8 (if applications to probability are not included, then one may omit Chapters 1 and 2 and parts of the other chapters).
- (iii) **Representation theory of finite groups** (possibly with applications to probability): Chapters 3, 4 (partially), 9, 10 and 11.

This book would never have been written without the encouragement and suggestions of Persi Diaconis. We thank him with deepest gratitude.

We are also grateful to Alessandro Figà Talamanca who first introduced us to Gelfand pairs and to the work of Diaconis.

We express our gratitude to Philippe Bougerol, Philippe Delsarte, Charles Dunkl, Adriano Garsia, Rostislav I. Grigorchuk, Gerard Letac,

Cambridge University Press

978-0-521-88336-8 - Harmonic Analysis on Finite Groups: Representation Theory,  
Gelfand Pairs and Markov Chains

Tullio Ceccherini-Silberstein, Fabio Scarabotti and Filippo Tolli

Frontmatter

[More information](#)

---

*Preface*

xiii

Arun Ram, Jan Saxl and Wolfgang Woess for their interest in our work and their encouragement.

We also acknowledge, with warmest thanks, the most precious careful reading of some parts of the book by Reza Bourquin and Pierre de la Harpe who pointed out several inaccuracies and suggested several changes and improvements on our expositions.

We finally express our deep gratitude to David Tranah, Peter Thompson, Bethan Jones and Mike Nugent from Cambridge University Press for their constant and kindest help at all the stages of the editing process.

Roma, 14 February 2007

TCS, FS and FT