CHAPTER 1

Basic Considerations

Since ancient times many researchers have devoted themselves to predicting and explaining how bodies move under the action of forces. This is the scope of the subject of dynamics, which consists of two phases: kinematics and kinetics. A kinematical analysis entails a quantitative description of the motion of bodies without concern for what is causing the motion. Sometimes that is all that is required, as would be the case if we needed to ascertain the output motion of a gear system or linkage. More significantly, a kinematical analysis will always be a key component of a kinetics study, which analyzes the interplay between forces and motion. Indeed, we will see that the kinematical description provides the skeleton on which the laws of kinetics are applied.

A primary objective will be the development of procedures for applying kinematics and kinetics principles in a logical and consistent manner, so that one may successfully analyze systems that have novel features. Particular emphasis will be placed on three-dimensional systems, some of which feature phenomena that are counterintuitive for those whose experience is limited to systems that move in a plane. A related objective is development of the capability to address realistic situations encountered in current engineering practice.

The scope of this text is limited to situations that are accurately described by the classical laws of physics. The only kinetics laws we will take to be axiomatic are those of Newton, which are accurate whenever the object of interest is moving much more slowly than the speed of light. Newton's Laws pertain only to a particle. The derivation of a variety of principles that extend these laws to bodies having significant dimensions will be treated in depth. We will limit our attention to systems in which all bodies may be considered to be particles or rigid bodies. The dynamics of flexible bodies, which is the subject of vibrations, is founded on the kinematics and kinetics concepts we will establish. We shall begin by reviewing the fundamental aspects of Newton's Laws. Although the reader is likely to have already studied these concepts, the intent is to provide a consistent foundation for later developments.

1.1 VECTOR OPERATIONS

1.1.1 Algebra and Computations

Almost every quantity of importance in dynamics is vectorial in nature. Such quantities have a direction in which they are oriented, as well as a magnitude. The kinematical vectors of primary importance for our initial studies are position, velocity, and acceleration, and the kinetics quantities are force and moment. Some quantities have magnitude and direction, but are not vectors. One example, which will play a major role in Chapter 3, is a finite rotation about an axis. An additional requirement for vector quantities is that they add according to the parallelogram law. This entails a graphical representation of vectors in which an arrow indicates the direction of the vector and the length of the arrow is proportional to the magnitude of the vector. A graphical representation of the summation operation is shown in Fig. 1.1(a), which shows that the addition of two vectors $\bar{A}$ and $\bar{B}$ may be constructed in either of two ways. Vectors $\bar{A}$ and $\bar{B}$ may be placed tail to tail, and then considered to form two sides of a parallelogram. Then $\bar{A} + \bar{B}$ is the main diagonal, with the sense defined to be from the common tail to the opposite corner. An alternative picture places the tail of $\bar{B}$ at the head of $\bar{A}$. The sum then extends from the tail of $\bar{A}$ to the head of $\bar{B}$.

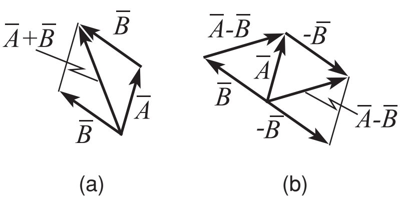


Figure 1.1. Diagrammatical construction of the sum and difference of two vectors.

An important aspect of these constructions is that a sum is independent of the sequence in which the vectors are added. This is the commutative property, which is stated as

$$\bar{A} + \bar{B} = \bar{B} + \bar{A}. \tag{1.1.1}$$

A diagram showing the sum of three vectors leads to the associative property,

$$\left(\bar{A} + \bar{B}\right) + \bar{C} = \bar{A} + \left(\bar{B} + \bar{C}\right). \tag{1.1.2}$$

Another important property comes from the observation that multiplying a vector by a scalar number does not affect its direction, but the magnitude is multiplied by that factor's absolute value, that is,

$$\left|\gamma \bar{A}\right| = |\gamma| \left|\bar{A}\right|. \tag{1.1.3}$$

A corollary of this property is that multiplying $\bar{A} + \bar{B}$ in Fig. 1.1(a) by a scalar changes the length of the diagonal, which requires that the individual sides be scaled by the same factor. Thus,

$$\gamma \left(\bar{A} + \bar{B}\right) = \gamma \bar{A} + \gamma \bar{B}, \tag{1.1.4}$$

which is the distributive property for vector addition.

If the γ factor in Eq. (1.1.3) is negative, $\gamma \bar{A}$ will be parallel to $\bar{A}$, but in the opposite sense. This observation leads to graphical rules for subtracting vectors. Multiplying a vector by -1 only reverses the sense of the vector. Because $\bar{A} - \bar{B} \equiv A + \left(-\bar{B}\right)$, the difference of two vectors may be constructed in one of three ways, as depicted graphically in Fig. 1.1(b). The difference may be formed by placing $\bar{A}$ and $-\bar{B}$ tail to tail, which forms a parallelogram. Then $\bar{A} - \bar{B}$ extends from the common tail to the opposite corner. A different rule leading to the same result comes from the observation that the parallelogram in Fig. 1.1(b) is identical to the one in Fig. 1.11(a). Thus the difference may be formed by placing $\bar{A}$ and $\bar{B}$ tail to tail, so that $\bar{A} - \bar{B}$ extends from the tip of $\bar{B}$ to the tip of $\bar{A}$. The third construction forms $\bar{A} - \bar{B}$ by placing the tail of $-B$ at the head of $\bar{A}$, in which case $\bar{A} - \bar{B}$ extends from the tail of $\bar{A}$ to the tip of $-\bar{B}$. Regardless of how one goes about forming the difference, it is wise to verify that forming $\bar{B} + \left(\bar{A} - \bar{B}\right)$ actually gives $\bar{A}$.

We will occasionally employ a diagrammatic approach to vector operations for derivations, but it is awkward and not easily implemented in mathematical software, especially for three-dimensional situations. Representation of vectorial quantities in component form addresses these issues. Let xyz denote a set of orthogonal Cartesian coordinates. Unit vectors $\bar{i}$, $\bar{j}$, and $\bar{k}$, whose magnitude is unity without dimensionality, are defined to be parallel to the x, y, and z axes, respectively. To represent its components, vector $\bar{A}$ in Fig. 1.2 has been situated with its tail at the origin of xyz.

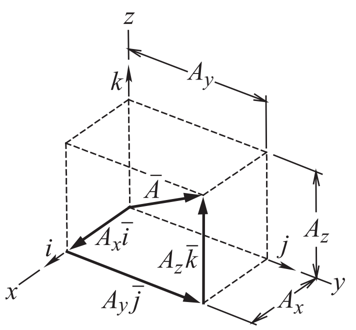


Figure 1.2. Unit vectors of a Cartesian coordinate system and the construction of vector components.

The edges of the box in the figure are constructed from the three lines that are perpendicular to a coordinate plane and intersect the tip of $\bar{A}$. The length of each line is the component of the vector, denoted with the subscript of the associated axis. (The length of a side would be the negative of the corresponding component's value if that side projected onto the negative coordinate axis.) Figure 1.2 shows that a vector along each edge of the box may be constructed by multiplying the component by the corresponding unit vector; see Eq. (1.1.3). The three such vectors depicted in the figure are situated head to tail, so their sum extends from the tail of the first, $A_x\bar{i}$, to the head of the third, $A_z\bar{k}$, but that is the original vector $\bar{A}$. Hence,

$$\boxed{\bar{A} = A_x\bar{i} + A_y\bar{j} + A_z\bar{k}.} \tag{1.1.5}$$

This is the *component representation of a vector*.

The utility of a component representation is that operations can be performed on the individual components without recourse to diagrams. By the Pythagorean theorem the magnitude of $\bar{A}$ is

$$\left|\bar{A}\right| = \left(A_x^2 + A_y^2 + A_z^2\right)^{1/2}. \tag{1.1.6}$$

In many situations we need to construct a unit vector parallel to a vector. This is readily obtained from the preceding equation as

$$\boxed{\bar{A} = \left|\bar{A}\right| \bar{e}_A \iff \bar{e}_A = \frac{\bar{A}}{\left|\bar{A}\right|}.} \tag{1.1.7}$$

The operations of adding or subtracting vectors are performed by operating on the individual components in accord with the properties in Eqs. (1.1.2) and (1.1.4):

$$\begin{aligned}\bar{A} \pm \bar{B} &= \left(A_x\bar{i} + A_y\bar{j} + A_z\bar{k}\right) \pm \left(B_x\bar{i} + B_y\bar{j} + B_z\bar{k}\right)\\ &= \left(A_x\bar{i} \pm B_x\bar{i}\right) + \left(A_y\bar{j} \pm B_y\bar{j}\right) + \left(A_z\bar{k} \pm B_z\bar{k}\right),\end{aligned}$$

$$\boxed{\bar{A} \pm \bar{B} = (A_x \pm B_x)\,\bar{i} + (A_y \pm B_y)\,\bar{j} + (A_z \pm B_z)\,\bar{k}.} \tag{1.1.8}$$

There are two types of products of two vectors. The *dot product* is also known as the *scalar product* because it is a scalar result. It is defined in terms of the angle ϕ between the vectors when they are placed tail to tail, according to

$$\bar{A} \cdot \bar{B} \equiv \left|\bar{A}\right| \left|\bar{B}\right| \cos\phi. \tag{1.1.9}$$

To avoid ambiguity, we limit the angle to $0 \le \phi \le \pi$. It is clear from this definition that the order in which a product is taken does not affect the result, so a dot product is commutative:

$$\bar{A} \cdot \bar{B} = \bar{B} \cdot \bar{A}. \tag{1.1.10}$$

One of the reasons why a dot product is useful is described by Fig. 1.3, where $\left|\bar{B}\right| \cos\phi$ is shown to be the projection of $\bar{B}$ in the direction of $\bar{A}$, in other words, the component of $\bar{B}$ in the direction of $\bar{A}$. That figure also shows that $\left|\bar{A}\right| \cos\phi$ is the component of $\bar{A}$ in the direction of $\bar{B}$. Thus a dot product may be interpreted to be the magnitude of one vector multiplied by the parallel component of the other vector. In the event where they form an obtuse angle, $\pi/2 < \phi \le \pi$, the dot product will be negative, meaning that the component is in the opposite sense from the vector on which it is projected.

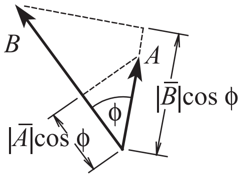


Figure 1.3. Dot product of two vectors, showing the component of each vector parallel to the other.

A dot product can be proven to be distributive, which may be stated as

$$\left(\alpha\bar{A} + \beta\bar{B}\right) \cdot \bar{C} = \alpha\bar{A} \cdot \bar{C} + \beta\bar{B} \cdot \bar{C}. \tag{1.1.11}$$

The significance of this property is that it enables us to evaluate a dot product directly in terms of the components of each vector. This comes about from the fact that $\bar{i}$, $\bar{j}$, and $\bar{k}$ are mutually orthogonal unit vectors, so that

$$\begin{aligned}&\bar{i} \cdot \bar{i} = \bar{j} \cdot \bar{j} = \bar{k} \cdot \bar{k} = 1,\\ &\bar{i} \cdot \bar{j} = \bar{j} \cdot \bar{i} = \bar{j} \cdot \bar{k} = \bar{k} \cdot \bar{j} = \bar{k} \cdot \bar{i} = \bar{i} \cdot \bar{k} = 0.\end{aligned} \tag{1.1.12}$$

Combining these fundamental dot products with Eq. (1.1.11) leads to evaluation of a dot product according to

$$\begin{aligned}\bar{A}\cdot\bar{B} &= \left(A_x\bar{i}+A_y\bar{j}+A_z\bar{k}\right)\cdot\left(B_x\bar{i}+B_y\bar{j}+B_z\bar{k}\right)\\ &= \left(A_x\bar{i}\right)\cdot\left(B_x\bar{i}\right)+\left(A_x\bar{i}\right)\cdot\left(B_y\bar{j}\right)+\left(A_x\bar{i}\right)\cdot\left(B_z\bar{k}\right)\\ &\quad+\left(A_y\bar{j}\right)\cdot\left(B_x\bar{i}\right)+\left(A_y\bar{j}\right)\cdot\left(B_y\bar{j}\right)+\left(A_y\bar{j}\right)\cdot\left(B_z\bar{k}\right)\\ &\quad+\left(A_z\bar{k}\right)\cdot\left(B_x\bar{i}\right)+\left(A_z\bar{k}\right)\cdot\left(B_y\bar{j}\right)+\left(A_z\bar{k}\right)\cdot\left(B_z\bar{k}\right),\end{aligned}$$

$$\boxed{\bar{A}\cdot\bar{B} = A_xB_x + A_yB_y + A_zB_z.} \tag{1.1.13}$$

A useful corollary of the preceding is that the length of a vector may be evaluated from a dot product,

$$\left|\bar{A}\right| = \sqrt{\bar{A}\cdot\bar{A}}. \tag{1.1.14}$$

The *cross product* of two vectors is also known as the *vector product*, because it is defined to be a vector in the direction perpendicular to the plane formed when the vectors are brought tail to tail. The magnitude of a cross product is defined as

$$\left|\bar{A}\times\bar{B}\right| = \left|\bar{A}\right|\left|\bar{B}\right|\sin\phi, \tag{1.1.15}$$

where ϕ is the angle between the vectors, as it is for the dot product. As shown in Fig. 1.4, $\left|\bar{B}\right|\sin\phi$ is the magnitude of the component of $\bar{B}$ perpendicular to $\bar{A}$, and $\left|\bar{A}\right|\sin\phi$ is the component of $\bar{A}$ perpendicular to $\bar{B}$. Thus the magnitude of a cross product may be interpreted as the magnitude of one vector multiplied by the perpendicular component of the other vector. Figure 1.4 also shows that the sense of the cross-product direction is determined by the right-hand rule, in which the vectors are brought tail to tail, and the fingers of the right-hand curl from the first vector to the second, as indicated by the curling arrow. The extended thumb then gives the orientation of the cross product, which would be out of the plane depicted by Fig. 1.4.

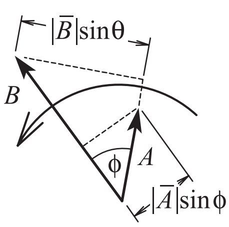


Figure 1.4. Construction of the cross product of two vectors showing the component of one vector perpendicular to the other. The curling arrow indicates the sense in which the fingers of the right hand should curl to form $\bar{A}\times\bar{B}$.

A cross product is not commutative because switching the sequence in which the product is formed reverses the sense of the curling arrow in Fig. 1.4. Thus,

$$\bar{B}\times\bar{A} = -\bar{A}\times\bar{B}. \tag{1.1.16}$$

The cross product does have the associative and distributive properties:

$$\begin{aligned}\left(\bar{A}\times\bar{B}\right)\times\bar{C} &= \bar{A}\times\left(\bar{B}\times\bar{C}\right),\\ \left(\alpha\bar{A}+\beta\bar{B}\right)\times\bar{C} &= \alpha\bar{A}\times\bar{C}+\beta\bar{B}\times\bar{C}.\end{aligned} \tag{1.1.17}$$

These properties lead to a rule for evaluating cross products in terms of vector components. We require that xyz be a right-handed coordinate system, so the fact that the unit vectors of the coordinate are mutually orthogonal gives

$$\begin{gathered}\bar{i}\times\bar{i}=\bar{j}\times\bar{j}=\bar{k}\times\bar{k}=0,\\ \bar{i}\times\bar{j}=\bar{k},\quad \bar{j}\times\bar{k}=\bar{i},\quad \bar{k}\times\bar{i}=\bar{j},\\ \bar{j}\times\bar{i}=-\bar{k},\quad \bar{k}\times\bar{j}=-\bar{i},\quad \bar{i}\times\bar{k}=-\bar{j}.\end{gathered} \tag{1.1.18}$$

A mnemonic device for remembering these products is to consider positive alphabetical order to proceed as $\bar{i}$ to $\bar{j}$ to $\bar{k}$, then back to $\bar{i}$. Applying these cross products in conjunction with the distributive law in Eqs. (1.1.17) leads to

$$\begin{aligned}\bar{A}\times\bar{B} &= \left(A_x\bar{i}+A_y\bar{j}+A_z\bar{k}\right)\times\left(B_x\bar{i}+B_y\bar{j}+B_z\bar{k}\right)\\ &= \left(A_x\bar{i}\right)\times\left(B_y\bar{j}\right)+\left(A_x\bar{i}\right)\times\left(B_z\bar{k}\right)+\left(A_y\bar{j}\right)\times\left(B_x\bar{i}\right)+\left(A_y\bar{j}\right)\times\left(B_z\bar{k}\right)\\ &\quad+\left(A_z\bar{k}\right)\times\left(B_x\bar{i}\right)+\left(A_z\bar{k}\right)\times\left(B_y\bar{j}\right)\\ &= A_xB_y\bar{k}-A_xB_z\bar{j}-A_yB_x\bar{k}+A_yB_z\bar{i}+A_zB_x\bar{j}-A_zB_y\bar{i},\end{aligned}$$

$$\boxed{\bar{A}\times\bar{B}=(A_yB_z-A_zB_y)\,\bar{i}+(A_zB_x-A_xB_z)\,\bar{j}+(A_xB_y-A_yB_x)\,\bar{k}.} \tag{1.1.19}$$

Some individuals, rather than carrying out a cross product term by term, as in the preceding evaluation, use a mnemonic device based on a determinant, specifically,

$$\bar{A}\times\bar{B}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}=\begin{vmatrix}A_x & A_y & A_z\\ B_x & B_y & B_z\\ \bar{i} & \bar{j} & \bar{k}\end{vmatrix}. \tag{1.1.20}$$

A common analytical approach we will encounter entails describing a vector in different ways and then equating the different descriptions. A component description of vectors enables us to convert the vector equality to a set of scalar equations, based on the fact that if two vectors are equal their like components must be equal. Thus,

$$\boxed{\bar{A}=\bar{B}\iff A_x=B_x,\quad A_y=B_y,\ \text{and}\ A_z=B_z.} \tag{1.1.21}$$

Position vectors are the fundamental kinematical quantities. In Fig. 1.5 the position vector extending from origin O to point P is labeled $\bar{r}_{P/O}$, which should be read as *the position vector to P from O*, or equivalently, *the position of P with respect to O*. Similarly, the position of point P with respect to point A is $\bar{r}_{P/A}$. The tail of $\bar{r}_{P/A}$ is situated at the head of $\bar{r}_{A/O}$, from which it follows that adding these vectors gives the position of point P with respect to point O:

$$\boxed{\bar{r}_{P/O}=\bar{r}_{A/O}+\bar{r}_{P/A}.} \tag{1.1.22}$$

This construction is fundamental to many operations in dynamics.

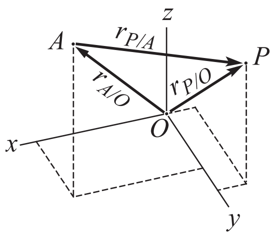


Figure 1.5. Observation of a moving point P by observers at points O and A.

The issue of how one carries out algebraic operations with vectors requires consideration of mathematical software. Three-dimensional vectors may be represented as matrices, which is the preferred data format for such popular programs as Matlab and Mathcad. Both programs allow one to carry out all vector operations using matrix notation. In Mathcad one proceeds by writing all vectors in matrix form and then carrying out the operations as indicated. For example, if $\bar{A} = 1\bar{i} + 2\bar{j} + 3\bar{k}$ and $\bar{B} = 3\bar{i} - \bar{j} - 5\bar{k}$, then the operation of constructing a unit vector parallel to $\bar{A} \times \bar{B}$, then verifying that this product is indeed perpendicular to $\bar{A}$ and $\bar{B}$, could proceed as

$$A := \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}, \quad B := \begin{Bmatrix} 3 \\ -1 \\ -5 \end{Bmatrix}, \quad C := A \times B, \quad e := \frac{C}{|C|}, \quad A_1 = A * e, \quad B_1 = B * e, \tag{1.1.23}$$

where := denotes Mathcad's equality operator, which is obtained by pressing the colon key, and the cross-product operator is obtained from the Ctrl-8 key combination. The dot product in matrix notation is obtained from the product of a three-element row matrix and a three-element column matrix, so one could evaluate the dot product in Mathcad by writing $A^{\mathrm{T}} * B$. An alternative is to simply multiply vectors to form a dot product, as was just done, which returns a scalar value.

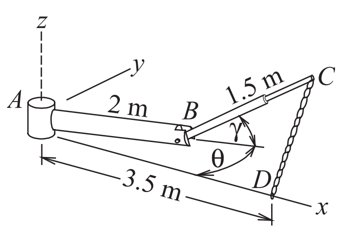
Matlab proceeds similarly. The cross product is implemented with the "cross" function; a dot product can be obtained from the "dot" function, or more simply as a standard row–column product. Thus, the preceding example could be carried out in Matlab as

```
A=[1  2  3]; B=[3  -1  -5]; C=cross(A,B);
    e=C/norm(C), A_1=A *e'; B_1=B*e';
```

Note that the "norm" without other arguments is Matlab's function for evaluating the (Euclidean) length of a vector. If one wishes, the preceding operations could be carried out with A and B defined to be three-element columns, for example, $A = [1;\ 2;\ 3]$. Other mathematical software programs have similar capabilities. Also, it is possible to implement these operations symbolically in some programs by use of matrix notation.

Ultimately, how one carries out computations is a matter of personal choice. The procedure used in this text generally will implement the operations term by term using the associative and distributive properties. One reason for this choice is that the notation is somewhat more compact. The second has to do with a common situation that will frequently arise, in which it will be necessary to combine vectors that are defined in terms of components relative to different coordinate systems. It is awkward to indicate which coordinate system a matrix is associated with, whereas the symbols used for unit vectors display that information unambiguously.

EXAMPLE 1.1 Robotic arm ABC induces a tensile force of 5000 N in cable CD. The orientation angles are $\theta = 25^\circ$ for link AB, which lies in the horizontal plane, and $\gamma = 40^\circ$ for rotation of arm BC. Pin B for this rotation is horizontal and perpendicular to arm AB, so AB and BC lie in a common vertical plane. Let $\bar{F}$ denote the force the cable exerts on the robotic arm. Determine (a) the component of $\bar{F}$ parallel to link BC, (b) the moment of $\bar{F}$ about end A, (c) the moment of $\bar{F}$ about the vertical z axis, and (d) the moment of $\bar{F}$ about arm AB.



Example 1.1

SOLUTION This example reviews some basic evaluations of force properties, which call for most of the standard vector operations. The cable is in tension, so it pulls the robotic arm from point C to point D. We express this as $\bar{F} = 5000\bar{e}_{D/C}$ N, where $\bar{e}_{D/C}$ is the notation we use for the *unit vector to D from C*. The first task is to determine the coordinates of point C, which we can find by constructing position vectors along arms AB and BC. We project point B onto the x and y axes to evaluate $\bar{r}_{B/A}$. Similarly, we project point C onto the xy plane, and then project that point onto the x and y axes. This gives

$$\bar{r}_{B/A} = 2\left(\cos\theta\bar{i} + \sin\theta\bar{j}\right) = 1.8126\bar{i} + 0.8452\bar{j} \text{ m},$$

$$\bar{r}_{C/B} = 1.5\cos\gamma\left(\cos\theta\bar{i} + \sin\theta\bar{j}\right) + 1.5\sin\gamma\bar{k} = 1.0414\bar{i} + 0.4856\bar{j} + 0.9642\bar{k}.$$

The desired position vector is the sum of these vectors:

$$\begin{aligned}\bar{r}_{C/A} &= \bar{r}_{B/A} + \bar{r}_{C/B} = (1.8126 + 1.0414)\,\bar{i} + (0.8452 + 0.4856)\,\bar{j} + (0 + 0.9642)\,\bar{k} \\ &= 2.8540\bar{i} + 1.3309\bar{j} + 0.9642\bar{k} \text{ m}.\end{aligned}$$

Because $\bar{r}_{C/A}$ and $\bar{r}_{D/A} = 3.5\bar{i}$ m are tail to tail, it follows that $\bar{r}_{D/C} = \bar{r}_{D/A} - \bar{r}_{C/A}$, which leads to $\bar{e}_{D/C}$ according to

$$\begin{aligned}\bar{e}_{D/C} &= \frac{\bar{r}_{D/A} - \bar{r}_{C/A}}{\left|\bar{r}_{D/A} - \bar{r}_{C/A}\right|} = \frac{0.6460\bar{i} - 1.3309\bar{j} - 0.9642\bar{k}}{(0.6460^2 + 1.3309^2 + 0.9642^2)^{1/2}} \\ &= 0.3658\bar{i} - 0.7537\bar{j} - 0.5460\bar{k}.\end{aligned}$$

Thus the force applied to the arm is

$$\bar{F} = 5000\bar{e}_{D/C} = 1829\bar{i} - 3768\bar{j} - 2730\bar{k} \text{ N.}$$

The component of $\bar{F}$ parallel to arm BC may be obtained from a dot product with the unit vector $\bar{e}_{C/B}$, which is readily constructed from $\bar{r}_{C/B}$, whose value has already been determined. Thus,

$$\begin{aligned} F_{BC} &= \bar{F} \cdot \bar{e}_{C/B} = \bar{F} \cdot \frac{\bar{r}_{C/B}}{\left|\bar{r}_{C/B}\right|} = \bar{F} \cdot \left(0.6943\bar{i} + 0.3237\bar{j} + 0.6428\bar{k}\right) \\ &= (1829)\,(0.6943) + (-3768)\,(0.3237) + (-2730)\,(0.6428) = -1705 \text{ N.} \end{aligned} \qquad \triangleleft$$

Negative F_{BC} indicates that the projection of $\bar{F}$ onto line BC is opposite the sense of $\bar{e}_{C/B}$.

The moment of a force may be evaluated from a cross product with a position vector from the reference point for the moment to the point where the force is applied. Hence,

$$\begin{aligned} \bar{M}_A &= \bar{r}_{C/A} \times \bar{F} = \left(2.8540\bar{i} + 1.3309\bar{j} + 0.9642\bar{k}\right) \times \left(1829\bar{i} - 3768\bar{j} - 2730\bar{k}\right) \\ &= (2.8540)\,(-3768)\,\bar{k} + (2.8540)\,(-2730)\left(-\bar{j}\right) + (1.3309)\,(1829)\left(-\bar{k}\right) \\ &\quad + (1.3309)\,(-2730)\,\bar{i} + (0.9642)\,(1829)\,\bar{j} + (0.9642)\,(-3768)\left(-\bar{i}\right) \\ &= 9555\bar{j} - 13189\bar{k} \text{ N-m.} \end{aligned} \qquad \triangleleft$$

The moment of a force about an axis may be determined by forming the moment about any point on that axis, and then evaluating the component of that moment in the direction of the axis. Thus the moment of $\bar{F}$ about the z axis is merely the $\bar{k}$ component of $\bar{M}_A$,

$$M_{Az} = \bar{M}_A \cdot \bar{k} = -13189 \text{ N-m.} \qquad \triangleleft$$

A negative value indicates that the sense of this moment is determined by aligning the extended thumb of the right hand in the $-\bar{k}$ direction. The same reasoning shows that the moment of $\bar{F}$ about arm AB is obtained with a dot product involving $\bar{e}_{B/A}$,

$$M_{AB} = \bar{M}_A \cdot \bar{e}_{B/A} = \bar{M}_A \cdot \frac{\bar{r}_{B/A}}{\left|\bar{r}_{B/A}\right|} = 4038 \text{ N-m.} \qquad \triangleleft$$

1.1.2 Vector Calculus—Velocity and Acceleration

The primary kinematical variables for our initial studies are position, velocity, and acceleration. Velocity is defined to be the time derivative of position, and acceleration is the time derivative of velocity, so we need to establish how to handle derivatives of vectors. Because time derivatives are performed frequently, it is standard notation to use an overdot to denote each such operation. Overbars are used here to indicate that a quantity is a vector; the reader is encouraged to use the same notation.

Most of the laws for calculus operations are the same as those for scalar variables. Their adaptation requires that vector quantities be indicated unambiguously. In the following, $\bar{A}$ and $\bar{B}$ are time-dependent vector functions, and α and β are scalar functions of time.

Definition of a derivative:

$$\frac{d\bar{A}}{dt} \equiv \dot{\bar{A}} \equiv \lim_{\Delta t \to 0} \frac{\bar{A}(t+\Delta t) - \bar{A}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\bar{A}(t) - \bar{A}(t-\Delta t)}{\Delta t}. \tag{1.1.24}$$

Definite integration:

$$\text{If } \dot{\bar{A}} = \bar{B}, \text{ then } \bar{B}(t) = B(t_0) + \int_{t_0}^{t} A(\tau)\, d\tau. \tag{1.1.25}$$

Derivative of a sum:

$$\frac{d}{dt}\left(\bar{A} + \bar{B}\right) = \dot{\bar{A}} + \dot{\bar{B}}. \tag{1.1.26}$$

Derivative of products:

$$\begin{aligned} \frac{d}{dt}\left(\alpha \bar{A}\right) &= \dot{\alpha}\bar{A} + \alpha\dot{\bar{A}}, \\ \frac{d}{dt}\left(\bar{A}\cdot\bar{B}\right) &= \dot{\bar{A}}\cdot\bar{B} + \bar{A}\cdot\dot{\bar{B}}, \\ \frac{d}{dt}\left(\bar{A}\times\bar{B}\right) &= \dot{\bar{A}}\times\bar{B} + \bar{A}\times\dot{\bar{B}}. \end{aligned} \tag{1.1.27}$$

As an immediate consequence of these properties, all calculus operations may be performed in terms of vector components. We consider here only situations in which xyz is a fixed coordinate system, so that $\bar{i}$, $\bar{j}$, and $\bar{k}$ are constant vectors, which means that $d\bar{i}/dt = d\bar{j}/dt = d\bar{k}/dt = 0$. We then find that

$$\frac{d\bar{A}}{dt} = \frac{d}{dt}\left(A_x\bar{i} + A_y\bar{j} + A_z\bar{k}\right) = \dot{A}_x\bar{i} + \dot{A}_y\bar{j} + \dot{A}_z\bar{k}. \tag{1.1.28}$$

A common situation that arises in many phases of our study of kinematics involves a vector that depends on some parameter α, which in turn varies with time. Differentiation of the vector with respect to time in this circumstance can be performed with the chain rule:

$$\frac{d\bar{A}}{dt} = \frac{d\bar{A}}{d\alpha}\frac{d\alpha}{dt} \equiv \dot{\alpha}\frac{d\bar{A}}{d\alpha}. \tag{1.1.29}$$

The chain rule may be extended by partial differentiation to situations in which the vector depends on two or more time-dependent parameters, according to

$$\frac{d\bar{A}}{dt} = \dot{\alpha}\frac{\partial\bar{A}}{\partial\alpha} + \dot{\beta}\frac{\partial\bar{A}}{\partial\beta} + \cdots. \tag{1.1.30}$$

In the present notation, where $\bar{r}_{P/O}$ denotes the position of point P with respect to the origin O of a fixed coordinate system, then the velocity and acceleration of that point are

$$\boxed{\bar{v} \equiv \dot{\bar{r}}_{P/O}, \quad \bar{a} \equiv \dot{\bar{v}} = \ddot{\bar{r}}_{P/O}.} \tag{1.1.31}$$