1
Introduction

This book is devoted to the theory of J-contractive and J-inner **mvf's** (matrix valued functions) and a number of its applications, where J is an $m \times m$ signature matrix, i.e., J is both unitary and self adjoint with respect to the standard inner product in $\mathbb{C}^m$. This theory plays a significant role in a number of diverse problems in mathematical systems and networks, control theory, stochastic processes, operator theory and classical analysis. In particular, it is an essential ingredient in the study of direct and inverse problems for canonical systems of integral and differential equations, since the matrizant (fundamental solution) $U_x(\lambda) = U(x, \lambda)$ of the canonical integral equation

$$u(x, \lambda) = u(0, \lambda) + i\lambda \int_0^x u(s, \lambda) dM(s) J, \quad 0 \leq x < d, \tag{1.1}$$

based on a nondecreasing $m \times m$ mvf $M(x)$ on the interval $0 \leq x < d$ is an entire mvf in the variable λ that is J-inner in the open upper half plane $\mathbb{C}_+$ for each point $x \in [0, d)$:

(1) $U_x(\lambda)$ is J-contractive in $\mathbb{C}_+$:

$$U_x(\lambda)^* J U_x(\lambda) \leq J \quad \text{for} \quad \lambda \in \mathbb{C}_+$$

and

(2) $U_x(\lambda)$ is J-unitary on the real axis $\mathbb{R}$:

$$U_x(\lambda)^* J U_x(\lambda) = J \quad \text{for} \quad \lambda \in \mathbb{R}.$$

Moreover, $U_x(\lambda)$ is monotone in the variable x in the sense that

$$U_{x_2}(\lambda)^* J U_{x_2}(\lambda) \leq U_{x_1}(\lambda)^* J U_{x_1}(\lambda) \quad \text{if} \quad 0 \leq x_1 \leq x_2 < d$$

and $\lambda \in \mathbb{C}_+$. These properties follow from the fact that the matrizant $U_x(\lambda) = U(x,\lambda)$ is a solution of the system (1.1) with $U_0(\lambda) = I_m$, i.e.,

$$U(x,\lambda) = I_m + i\lambda \int_0^x U(s,\lambda)dM(s)J, \quad 0 \le x < d,$$

and hence satisfies the identity

$$U_{x_2}(\lambda)JU_{x_2}(\omega)^* - U_{x_1}(\lambda)JU_{x_1}(\omega)^* = -i(\lambda - \overline{\omega}) \int_{x_1}^{x_2} U_x(\lambda)dM(x)U_x(\omega)^*.$$

The family $U_x(\lambda)$ is also continuous in the variable x and normalized by the condition

$$U_x(0) = I_m \quad \text{for} \quad 0 \le x < d.$$

The most commonly occuring signature matrices (except for $J = \pm I_m$) are the matrices

$$j_{pq} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}, \; J_p = \begin{bmatrix} 0 & -I_p \\ -I_p & 0 \end{bmatrix} \text{ and } \mathcal{J}_p = \begin{bmatrix} 0 & -iI_p \\ iI_p & 0 \end{bmatrix},$$

$-j_{pq}$, $-J_p$ and $-\mathcal{J}_p$. The equivalences

$$\begin{bmatrix} \varepsilon^* & I_q \end{bmatrix} j_{pq} \begin{bmatrix} \varepsilon \\ I_q \end{bmatrix} \le 0 \iff \varepsilon^*\varepsilon \le I_q;$$

$$\begin{bmatrix} \varepsilon^* & I_p \end{bmatrix} J_p \begin{bmatrix} \varepsilon \\ I_p \end{bmatrix} \le 0 \iff \varepsilon + \varepsilon^* \ge 0$$

and

$$\begin{bmatrix} \varepsilon^* & I_p \end{bmatrix} \mathcal{J}_p \begin{bmatrix} \varepsilon \\ I_p \end{bmatrix} \le 0 \iff \frac{\varepsilon - \varepsilon^*}{i} \ge 0$$

indicate a connection between the signature matrices j_{pq}, J_p and $\mathcal{J}_p$ and the classes

$$\begin{aligned} \mathcal{S}_{const}^{p\times q} &= \{\varepsilon \in \mathbb{C}^{p\times q} : \varepsilon^*\varepsilon \le I_q\} \quad \text{of contractive } p\times q \text{ matrices;} \\ \mathcal{C}_{const}^{p\times p} &= \{\varepsilon \in \mathbb{C}^{p\times p} : \varepsilon + \varepsilon^* \ge 0\} \quad \text{of positive real } p\times p \text{ matrices;} \\ i\mathcal{C}_{const}^{p\times p} &= \{\varepsilon \in \mathbb{C}^{p\times p} : (\varepsilon - \varepsilon^*)/i \ge 0\} \quad \text{of positive imaginary} \\ & \qquad p\times p \text{ matrices.} \end{aligned}$$

Moreover, if an $m \times m$ matrix U is J-contractive, i.e., if

$$U^*JU \leq J, \tag{1.2}$$

then the inequality

$$[x^* \quad I]J\begin{bmatrix} x \\ I \end{bmatrix} \leq 0 \tag{1.3}$$

implies that

$$[x^* \quad I]U^*JU\begin{bmatrix} x \\ I \end{bmatrix} \leq 0 \tag{1.4}$$

and hence, the linear fractional transformation

$$T_U[x] = (u_{11}x + u_{12})(u_{21}x + u_{22})^{-1}, \tag{1.5}$$

based on the appropriate four block decomposition of U, maps a matrix x in the class $\mathcal{F}_{const}(J)$ of matrices that satisfy the condition (1.3) into $\mathcal{F}_{const}(J)$, if x is admissible, i.e., if $\det(u_{21}x + u_{22}) \neq 0$.

Conversely, if $J \neq \pm I_m$ and U is an $m \times m$ matrix with $\det U \neq 0$ such that T_U maps admissible matrices $x \in \mathcal{F}_{const}(J)$ into $\mathcal{F}_{const}(J)$, then

$$\rho U \text{ is a } J\text{-contractive matrix for some } \rho \in C \setminus \{0\}. \tag{1.6}$$

Moreover, if T_U also maps (admissible) matrices x that satisfy (1.3) with equality into matrices with the same property, then the matrices ρU, considered in (1.6) are automatically J-unitary, i.e., $(\rho U)^*J(\rho U) = J$. These characterizations of the classes of J-contractive and J-unitary matrices are established in Chapter 2. The proofs are based on a number of results in the geometry of the space $\mathbb{C}^m$ with indefinite inner product

$$[\xi, \eta] = \eta^*J\xi$$

defined by an $m \times m$ signature matrix J, which are also presented in Chapter 2.

Analogous characterizations of the classes $\mathcal{P}(J)$ and $\mathcal{U}(J)$ of meromorphic J-contractive and J-inner mvf's in $\mathbb{C}_+$ are established in Chapter 4. These characterizations are due to L. A. Simakova. They are not simple corollaries of the corresponding algebraic results in Chapter 2: if the given $m \times m$ mvf $U(\lambda)$ is meromorphic in $\mathbb{C}_+$ with $\det U(\lambda) \not\equiv 0$ in $\mathbb{C}_+$ and $\rho(\lambda)U(\lambda) \in \mathcal{P}(J)$, then $\rho(\lambda)$ must be a meromorphic function in $\mathbb{C}_+$. To obtain such characterizations of mvf's in the classes $\mathcal{P}(J)$ and $\mathcal{U}(J)$ requires a number of results on inner-outer factorizations of scalar holomorphic functions in the Smirnov class $\mathcal{N}_+$ in $\mathbb{C}_+$ and inner denominators of scalar meromorphic functions in the Nevanlinna class $\mathcal{N}$ of functions with bounded characteristic in $\mathbb{C}_+$, and the Smirnov maximum principle in the class $\mathcal{N}_+$. This material and generalizations to $p \times q$ mvf's in the classes $\mathcal{N}_+^{p\times q}$ and $\mathcal{N}^{p\times q}$ with entries in the classes $\mathcal{N}_+$ and $\mathcal{N}$, respectively, is presented in Chapter 3. In particular, the Smirnov maximum principle, inner-outer factorization and a number of denominators for mvf's $f \in \mathcal{N}^{p\times q}$ are discussed in this chapter. Thus, Chapters 2 and 3 are devoted to topics in linear algebra and function theory for scalar and matrix valued functions that are needed to study J-contractive and J-inner mvf's as well as the other problems considered in the remaining chapters.

The sets $\mathcal{P}(J)$ and $\mathcal{U}(J)$ are multiplicative semigroups. In his fundamental paper [Po60] V. P. Potapov obtained a multiplicative representation for mvf's $U \in \mathcal{P}(J)$ with $\det U(\lambda) \not\equiv 0$ that is a far reaching generalization of the Blaschke-Riesz-Herglotz representation

$$u(\lambda) = b(\lambda)\exp\{i\alpha + i\beta\lambda\}\exp\left\{-\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{1+\mu\lambda}{\mu-\lambda}d\sigma(\mu)\right\} \tag{1.7}$$

of scalar holomorphic functions $u(\lambda)$ in $\mathbb{C}_+$ with $|u(\lambda)| \leq 1$. In formula (1.7) $b(\lambda)$ is a Blaschke product, $\alpha \in \mathbb{R}$, $\beta \geq 0$ and $\sigma(\mu)$ is a bounded nondecreasing function on $\mathbb{R}$. To obtain his multiplicative representation, Potapov used the factors that are now known as elementary Blaschke-Potapov factors. If $J \neq \pm I_m$, there are four kinds of such factors according to whether the pole is in the open lower half plane $\mathbb{C}_-$, in $\mathbb{C}_+$, in $\mathbb{R}$, or at ∞. He obtained criteria for the convergence of infinite products of normalized elementary factors that generalizes the Blaschke condition, using his theory of the J modulus. The Potapov **multiplicative representation** of mvf's $U \in \mathcal{P}(J)$ with $\det U(\lambda) \not\equiv 0$, leads to factorizations of U of the form

$$U(\lambda) = B(\lambda)U_1(\lambda)U_2(\lambda)U_3(\lambda),$$

where $B(\lambda)$ is a BP (Blaschke-Potapov) product of elementary factors, $U_1(\lambda)$ and $U_3(\lambda)$ are entire J-inner mvf's that admit a representation as a multiplicative integral that is a generalization of the second factor in (1.7), and $U_2(\lambda)$ is a holomorphic, J-contractive invertible mvf in $\mathbb{C}_+$ that admits a representation as a multiplicative integral that is a generalization of the third factor in (1.7).

In view of Potapov's theorem, every entire J-inner mvf $U(\lambda)$ with $U(0) = I_m$ admits a multiplicative integral representation

$$U(\lambda) = \overset{\curvearrowright}{\int_0^d} \exp\{i\lambda dM(x)J\}, \qquad (1.8)$$

where $M(x)$ is a nondecreasing $m \times m$ mvf on $[0, d]$. Moreover, $M(x)$ may be chosen so that $M(x)$ is absolutely continuous on $[0, d]$ with derivative $H(x) = M'(x) \geq 0$ normalized by the condition trace $H(x) = 1$ a.e. on $[0, d]$. But even under these last conditions, $H(x)$ is not uniquely defined by $U(\lambda)$, in general.

Multiplicative integrals were introduced in the theory of integral and differential equations by Volterra. In particular, the matrizant $U_x(\lambda)$ of the integral equation (1.1) may be written in the form of a multiplicative integral.

$$U_x(\lambda) = \overset{\curvearrowright}{\int_0^x} \exp\{i\lambda dM(s)J\}, \quad 0 \leq x < d, \qquad (1.9)$$

and if $d < \infty$ and $M(x)$ is bounded on $[0, d]$, then formula (1.8) coincides with formula (1.9) with $x = d$, and $U(\lambda) = U_d(\lambda)$ is the monodromy matrix of the system (1.1). Thus, in view of Potapov's theorem, every entire mvf $U \in U(J)$ with $U(0) = I_m$ may by interpreted as the monodromy matrix of a system of the form (1.1) on $[0, d]$.

A number of Potapov's results on finite and infinite BP products and on the multiplicative representation of mvf's in $\mathcal{P}(J)$ are presented in Chapter 4, sometimes without proof.

The problem of describing all normalized $m \times m$ mvf's $H(x) \geq 0$ in a differential system of the form

$$\frac{d}{dx}u(x, \lambda) = i\lambda u(x, \lambda)H(x)J \quad \text{a.e. on } [0, d] \qquad (1.10)$$

with a given monodromy matrix $U_d(\lambda) = U(\lambda)$ ($U \in \mathcal{E} \cap \mathcal{U}(J)$ and $U(0) = I_m$) is one of a number of inverse problems for systems of the form (1.10). The system (1.10) arises by applying the Fourier-Laplace transform

$$u(x, \lambda) = \int_0^\infty e^{i\lambda t} v(x, t)dt$$

to the solution $v(x,t)$ of the Cauchy problem

$$\begin{aligned} \frac{\partial v}{\partial x}(x,t) &= -\frac{\partial v}{\partial t}(x,t)H(x)J, \quad 0 \le x \le d,\ 0 \le t < \infty, \\ v(x,0) &= 0. \end{aligned} \tag{1.11}$$

Since $u(x,\lambda) = u(0,\lambda)U_x(\lambda)$, $0 \le x \le d$, the monodromy matrix $U_d(\lambda)$ is the transfer function of the system with distributed parameters on the interval $[0,d]$ specified by $H(x)$; with input $v(0,t)$, output $v(d,t)$ and state $v(x,t)$ at time t. Thus, the inverse monodromy problem is the problem of recovering the distributed parameters $H(x)$, $0 \le x \le d$, described by the evolution equation (1.11) from the transfer function of this system.

Potapov's theorem establishes the existence of a solution of the inverse monodromy problem. The uniqueness of the solution is established only under some extra conditions on $U(\lambda)$ or $H(x)$.

If $J = \pm I_m$ the Brodskii-Kisilevskii condition

$$\text{type } \{U(\lambda)\} = \text{type } \{\det U(\lambda)\}$$

on the exponential type of the entire mvf $U(\lambda)$ is necessary and sufficient for uniqueness. If $J \ne \pm I_m$, then the problem is much more complicated, even for $m = 2$.

A fundamental theorem of L. de Branges states that every entire $\mathcal{J}_1$-inner 2×2 mvf $U(\lambda)$ with $U(0) = I_2$ and the extra symmetry properties

$$\overline{U(-\overline{\lambda})} = U(\lambda) \quad \text{and} \quad \det U(\lambda) = 1$$

is the monodromy matrix of exactly one canonical differential system of the form (1.10) with $J = \mathcal{J}_1$ and with real, normalized Hamiltonian $H(x) \ge 0$ a.e. on $[0,d]$.

The Brodskii-Kisilevskii criteria was obtained in the sixties as a criteria for the unicellularity of a simple dissipative Volterra operator with a given characteristic mvf $U(\lambda)$.

Characteristic functions of nonselfadjoint (and nonunitary) operators were introduced in the 1940's by M. S. Livsic, who showed that these functions define the operator up to unitary equivalence under the assumption of simplicity and that they are J-contractive in $\mathbb{C}_+$ (in the open unit disc $\mathbb{D}$, respectively). Moreover, he discovered that to each invariant subspace of the operator there corresponds a divisor of the characteristic function and, to an ordered chain of invariant subspaces, there corresponds a triangular representation of the operator that generates a multiplicative representation of the characteristic function of the operator. Livsic also proposed a triangular model of the operator based on the multiplicative representation of the characteristic function. This was one of the main motivations for the development of the theory of multiplicative representations of J-contractive mvf's by V. P. Potapov.

L. de Branges obtained his uniqueness theorem and a number of other results in harmonic analysis, by consideration of the reproducing kernel Hilbert spaces of entire vvf's (vector valued functions) with reproducing kernels $K_\omega(\lambda)$ defined by the entire J-inner 2×2 mvf's $U(\lambda)$ by the formula

$$K_\omega(\lambda) = \frac{J - U(\lambda)JU(\omega)^*}{\rho_\omega(\lambda)}, \quad \text{where } \rho_\omega(\lambda) = -2\pi i(\lambda - \overline{\omega}).$$

The theory of RKHS's (reproducing kernel Hilbert spaces) with kernels of this form (and others) was developed by him, partially in collaboration with J. Rovnyak for $m \times m$ mvf's $U \in \mathcal{P}(J)$ for $m \geq 2$ and even for operator valued functions $U(\lambda)$.

A number of results on the spaces $\mathcal{H}(U)$ for $U \in \mathcal{P}(J)$, and for the de Branges spaces $\mathcal{B}(\mathfrak{E})$ are discussed in Chapter 5. In particular it is shown that if $U \in \mathcal{P}(J)$ and $\det U(\lambda) \not\equiv 0$, then the vvf's f in the corresponding RKHS $\mathcal{H}(U)$ are meromorphic in $\mathbb{C} \setminus \mathbb{R}$ with bounded Nevanlinna characteristic in both $\mathbb{C}_+$ and $\mathbb{C}_-$. Thus, every vvf $f \in \mathcal{H}(U)$ has nontangential boundary values

$$f_+(\mu) = \lim_{\nu \downarrow 0} f(\mu + i\nu) \quad \text{and} \quad \lim_{\nu \downarrow 0} f(\mu - i\nu) = f_-(\mu) \quad \text{a.e. on } \mathbb{R}.$$

Moreover,

$$U \in \mathcal{U}(J) \Longleftrightarrow f_+(\mu) = f_-(\mu) \quad \text{a.e. on } \mathbb{R} \quad \text{for every } f \in \mathcal{H}(U).$$

Connsequently, every $f \in \mathcal{H}(U)$ may be be identified with its boundary values if $U \in \mathcal{U}(J)$.

The space $\mathcal{H}(U)$ is R_α invariant with respect to the generalized backwards shift operator R_α that is defined by the formula

$$(R_\alpha f)(\lambda) = \frac{f(\lambda) - f(\alpha)}{\lambda - \alpha}, \quad \lambda \neq \alpha,$$

for points λ and α in the domain of holomorphy of $U(\lambda)$.

The subclasses $\mathcal{U}_S(J)$, $\mathcal{U}_{rR}(J)$, $\mathcal{U}_{rsR}(J)$, $\mathcal{U}_{\ell R}(J)$ and $\mathcal{U}_{\ell sR}(J)$ of singular, right regular, right strongly regular, left regular and left strongly regular

J-inner mvf's are introduced in Chapter 4 and are characterized in terms of the properties of the boundary values of vvf's from $\mathcal{H}(U)$ in Chapter 5: if $U \in \mathcal{U}(J)$, then

$$\begin{aligned} U \in \mathcal{U}_{rsR}(J) &\iff \mathcal{H}(U) \subset L_2^m, \\ U \in \mathcal{U}_{rR}(J) &\iff \mathcal{H}(U) \cap L_2^m \quad \text{is dense in} \quad \mathcal{H}(U), \\ U \in \mathcal{U}_S(J) &\iff \mathcal{H}(U) \cap L_2^m = \{0\}. \end{aligned}$$

Moreover, if $U \in \mathcal{U}(J)$, then the transposed mvf $U^\tau \in \mathcal{U}(J)$ and

$$U \in \mathcal{U}_{\ell R}(J) \iff U^\tau \in \mathcal{U}_{rR}(J) \quad \text{and} \quad U \in \mathcal{U}_{\ell sR}(J) \iff U^\tau \in \mathcal{U}_{rR}(J).$$

Furthermore, the following implications hold when $\omega \notin \mathbb{R}$:

$$U \in \mathcal{U}_{rsR}(J) \cup \mathcal{U}_{\ell sR}(J) \Longrightarrow \rho_\omega^{-1} U \in L_2^{m\times m} \Longrightarrow U \in \mathcal{U}_{rR}(J) \cap \mathcal{U}_{\ell R}(J).$$

There are a number of other characterizations of these classes. Thus, for example, in Chapter 4, an mvf $U \in \mathcal{U}(J)$ is said to belong to the class $U_S(J)$ of singular J-inner mvf's if it is an outer mvf in the Smirnov class $\mathcal{N}_+^{m\times m}$ in $\mathbb{C}_+$, i.e., if $U \in \mathcal{N}_+^{m\times m}$ and $U^{-1} \in \mathcal{N}_+^{m\times m}$. Then an mvf $U \in \mathcal{U}(J)$ is said to be right (resp., left) regular J-inner, if it does not have a nonconstant right (resp., left) divisor in the multiplicative semigroup $\mathcal{U}(J)$ that belongs to $\mathcal{U}_S(J)$. Characterizations of the subclasses $\mathcal{U}_{rsR}(J)$ and $\mathcal{U}_{\ell sR}(J)$ in terms of the Treil-Volberg matricial version of the Muckenhoupt (A_2)-condition are established in Chapter 10.

Every mvf $U \in \mathcal{U}(J)$ admits a pair of essentially unique factorizations:

$$U(\lambda) = U_1(\lambda)U_2(\lambda), \quad \text{where } U_1 \in \mathcal{U}_{rR}(J) \text{ and } \mathcal{U}_2 \in U_S(J), \tag{1.12}$$

and

$$U(\lambda) = U_3(\lambda)U_4(\lambda), \quad \text{where } U_4 \in \mathcal{U}_{\ell R}(J) \text{ and } U_3 \in \mathcal{U}_S(J).$$

The second factorization follows from the first (applied to the transposed mvf's $U^\tau(\lambda)$). The first factorization formula is established in Chapter 7 by considering the connection between mvf's $W \in \mathcal{U}_{rR}(j_{pq})$ and the GSIP (generalized Schur interpolation problem) in the class

$$\mathcal{S}^{p\times q} = \{s \in H_\infty^{p\times q} : \|s\|_\infty \leq 1\},$$

where $H_\infty^{p\times q}$ is the Hardy space of holomorphic bounded $p \times q$ mvf's in $\mathbb{C}_+$. In this problem, three mvf's are specified: $s^\circ \in \mathcal{S}^{p\times q}$ and two inner mvf's

$b_1 \in \mathcal{S}^{p\times p}$ and $b_2 \in \mathcal{S}^{q\times q}$ and

$$\mathcal{S}(b_1, b_2; s^\circ) = \{s \in \mathcal{S}^{p\times q} : b_2^{-1}(s - s^\circ)b_2^{-1} \in H_\infty^{p\times q}\}$$

is the set of solutions to this problem. The GSIP based on s°, b_1 and b_2 is said to be completely indeterminate if, for every nonzero vector $\xi \in \mathbb{C}^q$, there exists an mvf $s \in \mathcal{S}(b_1, b_2; s^\circ)$ such that $s(\lambda)\xi \not\equiv s^\circ(\lambda)\xi$. An mvf $W \in \mathcal{U}(j_{pq})$ is the resolvent matrix of this GSIP if

$$\mathcal{S}(b_1, b_2; s^\circ) = \{T_W[\varepsilon] : \varepsilon \in \mathcal{S}^{p\times q}\}. \tag{1.13}$$

There are infinitely many resolvent matrices $W \in \mathcal{U}(j_{pq})$ for each completely indeterminate GSIP (a description is furnished in Chapter 7) and every such W automatically belongs to the class $\mathcal{U}_{rR}(j_{pq})$. Conversely, every mvf $W \in \mathcal{U}_{rR}(j_{pq})$ is the resolvent matrix of a completely indeterminate GSIP. The correspondence between the class $\mathcal{U}_{rR}(j_{pq})$ and the completely indeterminate GSIP's is established in Chapter 7. Moreover, $W \in \mathcal{U}_{rsR}(j_{pq})$ if and only if W is the resolvent matrix of a strictly completely indeterminate GSIP; i.e., if and only if there exists at least one $\varepsilon \in \mathcal{S}^{p\times q}$ such that $\|T_W[\varepsilon]\|_\infty < 1$. The correspondence between the subclasses $\mathcal{U}_{rR}(J_p)$ and $\mathcal{U}_{rsR}(J_p)$ and completely indeterminate and strictly completely indeterminate GCIP's (generalized Carathéodory interpolation problems) are discussed in Chapter 7 too. This chapter also contains formulas for resolvent matrices $U(\lambda)$ that are obtained from the formulas in Chapter 5 for $U \in \mathcal{U}_{rsR}(J)$ with $J = j_{pq}$ and $J = J_p$ from the description of the corresponding RKHS's $\mathcal{H}(U)$.

The results on GCIP's that are obtained in Chapter 7 are used in Chapter 8 to study bitangential generalizations of the Krein extension problem of extending a continuous mvf $g(t)$, given on the interval $-a \le t \le a$, with a kernel

$$k(t, s) = g(t + s) - g(t) - g(-s) + g(0)$$

that is positive on $[0, a] \times [0, a]$ to a continuous mvf $\widetilde{g}(t)$ on $\mathbb{R}$ which is subject to analogous constraints on $[0, \infty) \times [0, \infty)$. In particular, the classes of entire mvf's U in $\mathcal{U}_{rR}(J_p)$ and $\mathcal{U}_{rSR}(J_p)$ are identified as the classes of resolvent matrices of completely indeterminate and strictly completely indeterminate bitangential extension problems for mvf's $g(t)$. A bitangential generalization of Krein's extension problem for continuous positive definite mvf's and Krein's extension problem for accelerants and the resolvent matrices for these problems are also considered in this chapter.

In Chapter 11 extremal values of entropy functionals for completely indeterminate generalized interpolation and extension problems are established in a uniform way that is based on the parametrizations of j_{pq} and J_p inner mvf's that was discussed in earlier chapters.

Every mvf $U \in \mathcal{U}(J)$ has a pseudocontinuation from $\mathbb{C}_+$ into $\mathbb{C}_-$ that is a meromorphic mvf of Nevanlinna class in $\mathbb{C}_-$. Consequently, every submatrix $s \in \mathcal{S}^{p\times q}$ of an inner mvf $S \in \mathcal{S}^{m\times m}$ admits such an extension to $\mathbb{C}_-$, as do mvf's of the form $s = T_W[\varepsilon]$ and and $c = T_A[\tau]$, where $W \in \mathcal{U}(j_{pq})$, $A \in \mathcal{U}(J_p)$, ε is a constant $p \times q$ contractive matrix and τ is a constant $p \times p$ matrix with $\tau + \tau^* \geq 0$. Such representations of the mvf's s and c arose in the synthesis of passive linear networks with losses by a lossless system with a scattering matrix S, a chain scattering matrix W or a transmission matrix A, repectively. The representations of s as a block of an $n \times n$ inner mvf S and $s = T_W[\varepsilon]$ and $c = T_A[\tau]$ with constant matrices $\varepsilon \in \mathcal{S}^{p\times q}$ and $\tau \in \mathcal{C}^{p\times p}$, respectively, are called Darlington representations, even though Darlington only worked with scalar rational functions $c \in \mathcal{C}$, and the scattering formalism described above was introduced by Belevich for rational mvf's $s \in \mathcal{S}^{p\times q}$. In the early seventies Darlington representations for mvf's $s \in \mathcal{S}^{p\times q}$ and $c \in \mathcal{C}^{p\times p}$ that admit pseudocontinuations into $\mathbb{C}_-$ were obtained independently by D. Z. Arov [Ar71] and P. Dewilde [De71]; generalizations to operator valued functions were obtained in [Ar71] and [Ar74a] and by R. Douglas and J. W. Helton in [DoH73]. Descriptions of the sets of representations and solutions of other inverse problems for J-inner mvf's are discussed in Chapter 9, which includes more detailed references.

In the study of bitangential interpolation problems and bitangential inverse problems for canonical systems, a significant role is played by a set $ap(W)$ of pairs $\{b_1, b_2\}$ of inner mvf's $b_1 \in \mathcal{S}^{p\times p}$ and $b_2 \in \mathcal{S}^{q\times q}$ that are associated with each mvf $W \in \mathcal{U}(j_{pq})$ and a set $ap_{II}(A)$ of pairs $\{b_3, b_4\}$ of $p \times p$ inner mvf's that is associated with each mvf $A \in \mathcal{U}(J_p)$. The inner mvf's in $\{b_1, b_2\}$ are defined in terms of the blocks w_{11} and w_{22} of W by the inner-outer factorization of $(w_{11}^{\#})^{-1} = (w_{11}(\overline{\lambda})^*)^{-1}$, which belongs to $\mathcal{S}^{p\times p}$ and the outer-inner factorization of w_{22}^{-1}, which belongs to $\mathcal{S}^{q\times q}$:

$$(w_{11}^{\#})^{-1} = b_1\varphi_1 \quad \text{and} \quad w_{22}^{-1} = \varphi_2 b_2.$$

The pair $\{b_3, b_4\} \in ap_{II}(A)$ is defined analogously in terms of the entries in the blocks of the de Branges matrix

$$\mathfrak{E}(\lambda) = \begin{bmatrix} E_-(\lambda) & E_+(\lambda) \end{bmatrix} = \begin{bmatrix} a_{22}(\lambda) - a_{21}(\lambda) & a_{22}(\lambda) + a_{21}(\lambda) \end{bmatrix}$$

that is defined in terms of the bottom blocks of A via the inner-outer and outer-inner factorizations of $(E_-^{\#}(\lambda))^{-1} = (E_-(\overline{\lambda})^*)^{-1}$ and $E_+(\lambda)^{-1}$ in the Smirnov class $\mathcal{N}_+^{p\times p}$:

$$(E_-^{\#})^{-1} = b_3\varphi_3 \quad \text{and} \quad E_+^{-1} = \varphi_4 b_4.$$

If the mvf A is holomorphic at the point $\lambda = 0$, then b_3 and b_4 are also holomorphic at the point $\lambda = 0$ and may be uniquely specified by imposing the normalization conditions $b_3(0) = I_p$ and $b_4(0) = I_p$.

To illustrate the role of associated pairs we first consider a system of the form (1.1) or (1.10) with $J = j_{pq}$. Then the matrizant W_x, $0 \le x < d$ is a monotonic continuous chain (with respect to the variable x) of entire j_{pq}-inner mvf's that is normalized by the condition $W_x(0) = I_m$. Correspondingly there is a unique chain of associated pairs $\{b_1^x(\lambda), b_2^x(\lambda)\}$ of entire inner mvf's with $b_1^x(0) = I_p$ and $b_2^x(0) = I_q$, and this chain is monotonic and continuous with respect to the variable x.

The class $\mathcal{U}_{rsR}(J)$ plays a significant role in a number of inverse problems for canonical systems of the forms (1.1) and (1.10). In particular, the matrizant $U_x(\lambda), 0 \le x < d$, of every canonical system that can be reduced to a Dirac system with locally summable potential belongs to the class $\mathcal{U}_{rsR}(J)$ for every $x \in [0, d)$; see e.g., [ArD05c], which includes applications to matrix Schrödinger equations with potentials of the form $q(x) = v^2(x) \pm v'(x)$ (even though the matrizant of the Schrödinger equation belongs to the class $\mathcal{U}_S(J)$).

In the authors' formulation of bitangential inverse problems, the given data is a monotonic continuous chain of pairs $\{b_1^x(\lambda), b_2^x(\lambda)\}$, $0 \le x < d$, and a spectral characteristic (e.g., a monodromy matrix, an input scattering or impedance matrix, or a spectral function) and the problem is to find a system with the given spectral characteristic that satisfies the two restrictions:

(1) $W_x \in \mathcal{U}_{rR}(j_{pq})$ for every $x \in [0, d)$.

(2) $\{b_1^x, b_2^x\} \in ap(W)$ for every $x \in [0, d)$.

These inverse problems were solved by Krein's method, which is based on identifying the matrizant with a family of resolvent matrices of an appropriately defined completely indeterminate extension problem; see e.g., [ArD05b], [ArD05c] and [ArD07b].

The Krein method works because for each completely indeterminate GSIP with given data b_1, b_2, s°, there is an mvf $W \in \mathcal{U}(j_{pq})$ such that (1.13) holds and $\{b_1, b_2\} \in ap(W)$ that is unique up to a right constant j_{pq} unitary multiplier. Moreover, if b_1 and b_2 are holomorphic at the point $\lambda = 0$, then W is holomorphic at the point $\lambda = 0$ and then may be uniquely specified by imposing the normalization $W(0) = I_m$. Furthermore, $W(\lambda)$ is entire if b_1 and b_2 are entire. These relationships are discussed in Chapters 7 and 8.

Descriptions of the RKHS's $\mathcal{H}(W)$ and $\mathcal{H}(A)$ based on associated pairs are discussed in Chapter 5.

The theory of the RKHS' $\mathcal{H}(U)$ and $\mathcal{B}(\mathfrak{E})$ is developed further and applied to construct functional models for Livsic-Brodskii operator nodes in Chapter 6. In this chapter the mvf's $U \in \mathcal{U}(J)$ that are holomorphic and normalized at the point $\lambda = 0$ (and in the even more general class $\mathcal{LB}(J)$) are identified as characteristic mvf's of Livsic-Brodskii nodes. Connections with conservative and passive linear continuous time invariant systems are also discussed.

Necessary and sufficient conditions for the characteristic mvf of a simple Livsic-Brodskii node to belong to the class $\mathcal{U}_{rsR}(J)$ are furnished in Chapter 10, and functional models of these nodes are given in terms of the associated pairs of the first and second kind of the characteristic function U of the node.

An $m \times m$ mvf $U \in \mathcal{P}(J)$ may be interpreted as the resolvent matrix of a symmetric operator with deficiency indices (m, m) in a Hilbert space. This theory was developed and applied to a number of problems in analysis by M. G. Krein; see e.g., Krein [Kr49] and the monograph [GoGo97]. The latter focuses on entire symmetric operators and, correspondingly, entire resolvent mvf's $U \in \mathcal{U}(J)$. Connections between the Krein theory of resolvent matrices and and characteristic mvf's of Livsic-Brodskii J-nodes with the de Branges theory of RKHS' $\mathcal{H}(U)$ were considered in [AlD84] and [AlD85]. Resolvent matrices of symmetric operators were identified as characteristic mvf's of generalized LB J-nodes by M. G. Krein and S. N. Saakjan [KrS70], A. V. Shtraus [Sht60], E. R. Tsekanovskii and Yu. L. Shmulyan [TsS77] and others.

An $m \times m$ mvf $U \in \mathcal{P}(J)$ may also be interpreted as the resolvent matrix of a completely indeterminate commutant lifting problem; see e.g., [SzNF70] and [FoFr90].

Finally, we remark that although we have chosen to focus on the classes $\mathcal{P}(J)$ and $\mathcal{U}(J)$ for the open upper half plane $\mathbb{C}_+$, most of the considered results have natural analogues for the open unit disc $\mathbb{D}$ with boundary $\mathbb{T}$.