The Mathematics of Logic

A guide to completeness theorems and their applications

This textbook covers the key material for a typical first course in logic for undergraduates or first year graduate students, in particular, presenting a full mathematical account of the most important result in logic: the Completeness Theorem for first-order logic.

Looking at a series of interesting systems increasing in complexity, then proving and discussing the Completeness Theorem for each, the author ensures that the number of new concepts to be absorbed at each stage is manageable, whilst providing lively mathematical applications throughout. Unfamiliar terminology is kept to a minimum; no background in formal set-theory is required; and the book contains proofs of all the required set theoretical results.

The reader is taken on a journey starting with König's Lemma, and progressing via order relations, Zorn's Lemma, Boolean algebras, and propositional logic, to Completeness and Compactness of first-order logic. As applications of the work on first-order logic, two final chapters provide introductions to model theory and non-standard analysis.

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Preface

Mathematical logic has been in existence as a recognised branch of mathematics for over a hundred years. Its methods and theorems have shown their applicability not just to philosophical studies in the foundations of mathematics (perhaps their original *raison d'être*) but also to 'mainstream mathematics' itself, such as the infinitesimal analysis of Abraham Robinson, or the more recent applications of model theory to algebra and algebraic geometry.

Nevertheless, these logical techniques are still regarded as somewhat 'difficult' to teach, and possibly rather unrewarding to the serious mathematician. In part, this is because of the notation and terminology that still survives as a relic of the original reason for the subject, and also because of the off-putting and didactically unnecessary logical precision insisted on by some of the authors of the standard undergraduate textbooks. This is coupled by the professional mathematician's very reasonable distrust of so much emphasis on 'inessential' non-mathematical details when he or she only requires an insight into the mathematics behind it and straightforward statements of the main mathematical results.

This book presents the material usually treated in a first course in logic, but in a way that should appeal to a suspicious mathematician wanting to see some genuine mathematical applications. It is written at a level suitable for an undergraduate, but with additional optional sections at the end of each chapter that contain further material for more advanced or adventurous readers. The core material in this book assumes as prerequisites only: basic knowledge of pure mathematics such as undergraduate algebra and real analysis; an interest in mathematics; and a willingness to discover and learn new mathematical material. The main goal is an understanding of the mathematical content of the Completeness Theorem for first-order logic, including some of its mathematically more interesting applications. The optional sections often require additional background material and more 'mathematical maturity' and go beyond a typical first undergraduate course. They may be of interest to beginning post-graduates and others.

The intended readership of this book is mathematicians of all ages and persuasions, starting at third year undergraduate level. Indeed, the 'unstarred' sections of this book form the basis of a course I have given at Birmingham University for third and fourth year students. Such a reader will want a good grounding in the subject, and a good idea of its scope and applications, but in general does not require a highly detailed and technical treatment.

On the other hand, for a full mathematical appreciation of what the Completeness Theorem has to offer, a detailed discussion of some naive set theory, especially Zorn's Lemma and cardinal arithmetic, is essential, and I make no apology for including these in some detail in this book.

This book is unusual, however, since I do not present the main concepts and goals of first-order logic straight away. Instead, I start by showing what the main mathematical idea of 'a completeness theorem' is, with some illustrations that have real mathematical content. The emphasis is on the content and possible applications of such completeness theorems, and tries to draw on the reader's mathematical knowledge and experience rather than any conception (or misconception) of what 'logic' is.

It seems that 'logic' means many things to different people, from puzzles that can be bought at a newsagent's shop, to syllogisms, arguments using Venn diagrams, all the way to quite sophisticated set theory. To prepare the reader and summarise the idea of a completeness theorem here, I should say a little about how I regard 'logic'.

The principal feature of logic is that it should be about reasoning or deduction, and should attempt to provide rules for valid inferences. If these rules are sufficiently precisely defined (and they should be), they become rules for manipulating strings of symbols on a page. The next stage is to attach meaning to these strings of symbols and try to present mathematical justification for the inference rules. Typically, two separate theorems are presented: the first is a 'Soundness Theorem' that says that *no incorrect deductions* can be made from the inference rules (where 'correct' means in terms of the meanings we are considering); the second is a 'Completeness Theorem' which says that *all correct deductions* that can be expressed in the system can actually be made using a combination of the inference rules provided. Both of these are precise mathematical theorems. Soundness is typically the more straightforward of the two to prove; the proof of completeness is usually much more sophisticated. Typically, it requires mathematical techniques that enable one to create a new mathematical 'structure' which shows that a particular supposed deduction that is not derivable in the system is not in fact correct.

Thus logic is not only about such connectives as 'and' and 'or', though the main systems, including propositional and first-order logic, do have symbols for these connectives. The power of the logical technique for the mathematician arises from the way the formal system of deduction can help organise a complex set of conditions that might be required in a mathematical construction or proof. The Completeness Theorem becomes a very general and powerful way of building interesting mathematical structures. A typical example is the application of first-order logic to construct number systems with infinitesimals that can used rigorously to present real calculus. This is the so-called nonstandard analysis of Abraham Robinson, and is presented in the last chapter of this book.

The mathematical content of completeness and soundness is well illustrated by König's Lemma on infinite finitely branching trees, and in the first chapter I discuss this. This is intended as a warm-up for the more difficult mathematics to come, and is a key example that I refer back to throughout the book.

Zorn's Lemma is essential for all the work in this book. I believe that by final year level, students should be starting to master straightforward applications of Zorn's Lemma. This is the main topic in Chapter 2. I do not shy away from the details, in particular giving a careful proof of Zorn's Lemma for countable posets, though the details of how Zorn's Lemma turns out to be equivalent to the Axiom of Choice is left for an optional section.

The idea of a formal system and derivations is introduced in Chapter 3, with a system based on strings of 0s and 1s that turns out to be closely related to König's Lemma. In the lecture theatre or classroom, I find this chapter to be particularly important and useful material, as it provides essential motivation for the Soundness Theorem. Given a comparatively simple system such as this, and asked whether a particular string σ can be derived from a set of assumptions Σ, students are all too ready to answer 'no' without justification. Where justification is offered, it is often of the kind, 'I tried to find a formal proof and this was my attempt, but it does not work.' So the idea of a careful proof by induction on the length of a formal derivation (and a carefully selected induction hypothesis) can be introduced and discussed without the additional complication of a long list of deduction rules to consider. The idea of semantics, and the Soundness and Completeness Theorems, arises from an investigation of general methods to show that certain derivations are not possible, and, to illustrate their power, König's Lemma is re-derived from the Soundness and Completeness Theorems for this system.

The reader will find systems with mathematically familiar derivations for the first time in Chapter 4. Building on previous material on posets, I develop a system for derivations about a poset, including rules such as 'if $a < b$ and

$b < c$ then $a < c$'. The system also has a way of expressing statements of the form 'a is not less than b', and this is handled using a Reductio Ad Absurdum Rule, a rule that is used throughout the rest of the book. By this stage, it should be clear what the right questions to ask about the system are, and the mathematical significance of the Completeness Theorem (the construction of a suitable partial order on a set) is clear. As a bonus, two pretty applications are provided: that any partial order can be 'linearised'; and that from a set of 'positive' assumptions a 'negative' conclusion can always be strengthened to a 'positive' one.

The material normally found in a more traditional course on mathematical logic starts with Chapter 5. Chapters 5 to 8 discuss boolean algebras and propositional logic. My proof system for propositional logic is chosen to be a form of natural deduction, but set out in a linear form on the page with clearly delineated 'subproofs' rather than a tree structure. This seems to be closest to a student's conception of a proof, and also provides clear proof strategies so that exercises in writing proofs can be given in a helpful and relatively painless way. (I emphasise the word 'relatively'. For most students, this aspect of logic is never painless, but at least the system clearly relates to informal proofs they might have written in other areas of mathematics.) I do not avoid explaining the precise connections between propositional logic and boolean algebra; these are important and elegant ideas, and are accessible to undergraduates who should be able to appreciate the analogies with algebra, especially rings and fields. More advanced students will also appreciate the fact that deep results such as Tychonov's Theorem and Stone Duality are only a few pages extra in an optional section. However, if time is short, the chapter on filters and ideals can be omitted entirely.

Chapters 9 and 10 are the central ones that cover first-order logic and the main Completeness Theorem. Apart from the choice of formal system (a development of the natural deduction system already used for propositional logic) they follow the usual pattern. These chapters are the longest in the book and will be found to be the most challenging so I have deliberately avoided many of the technically tricky issues such as: unique readability; the formal definition of the scope of a quantifier; or when a variable may be substituted by a term. An intelligent reader at this level using his or her basic mathematical training and intuition and following the examples is sure to do the 'right thing' and does not want to be bogged down in formal syntactic details. These technical details are of course important later on if one becomes involved in formalising logic in a first-order system such as set theory or arithmetic. But the place for that sort of work is certainly not a *first* course in logic. For those readers that need it, further details are available on the companion web-pages.

The method of proof of the Completeness Theorem is by 'Henkinising' the language and then using Zorn's Lemma to find a maximal consistent set of sentences. This is easier to describe to first-timers than tree-constructions of sets of consistent sentences with their required inductive properties, but is just as general and applicable. Two bonus optional sections for adventurous students with background in point-set topology include a topological view of the Compactness Theorem, and a proof of the full statement of the Omitting Types Theorem via Baire's Theorem, which is proved where needed.

Chapters 11 and 12 (which are independent of each other) provide applications of first-order logic. Chapter 11 presents an introduction to model theory, including the Löwenheim–Skolem Theorems, and (to put these in context) a short survey of categoricity, including a description of Morley's Theorem. This chapter is where infinite cardinals and cardinal arithmetic are used for the first time, and I carefully state all the required ideas and results before using them. Full proofs of these results are given in an optional section, using Zorn's Lemma only. The traditional options of using ordinals or the well-ordering principle are avoided as being likely to beg more questions than they answer to students without any prior knowledge in formal set theory. Chapter 12 presents an introduction to nonstandard analysis, including a proof of the Peano Existence Theorem on first-order differential equations. My presentation of nonstandard analysis is chosen to illustrate the main results of first-order logic and the interplay between the standard and nonstandard worlds, rather than to be optimal for fast proofs of classical results by nonstandard methods.

I have enjoyed writing this book and teaching from it. The material here is, to my mind, much more exciting and varied than the standard texts I learnt from as an undergraduate, and responses from the students who were given preliminary versions of these notes were good too. I can only hope that you, the reader, will derive a similar amount of pleasure from this book.

How to read this book

Chapters are deliberately kept as short as possible and discuss a single mathematical point. The chapters are divided into sections. The first section of each chapter is essential reading for all. The second section generally contains further applications, examples and exercises to test and expand on material presented in the previous section, and is again essential to read and explore. One or more extra ‘starred’ sections are then added to provide further commentary on the key material of the chapter and develop the material. These other sections are not essential reading and are intended for more inquisitive, ambitious or advanced readers with the background knowledge required. Chapter 8 may be omitted if time is short, and Chapters 11 and 12 are independent of each other.

Mathematical terminology is standard or explained in the text. Bold face entries in the index refer to definitions in the text; other entries provide further information on the term in question.

Additional material, including some technical definitions that I have chosen to omit in the printed text for the sake of clarity, further exercises, discussion, and some hints or answers to the exercises here, will be found on the companion web-site at http://web.mat.bham.ac.uk/R.W.Kaye/logic.