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Cambridge University Press
978-0-521-88218-7 - Elements of the Representation Theory of Associative Algebras: Volume 3: Representation-Infinite Tilted Algebras: London Mathematical Society Student Texts 72
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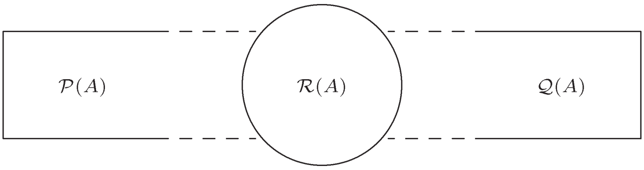
## Chapter XV

# Tubular extensions and tubular coextensions of algebras

In Volume 2, we study in detail the indecomposable modules and the shape of the Auslander–Reiten quiver $\Gamma(\text{mod } B)$ of concealed algebras of Euclidean type, that is, the tilted algebras $B$ of the form

$$B = \text{End } T_{KQ},$$

where $KQ$ is a hereditary algebra of Euclidean type and $T_{KQ}$ is a post-projective tilting $KQ$-module. We recall that every concealed algebra $B$ of Euclidean type is representation-infinite and the Auslander–Reiten quiver $\Gamma(\text{mod } B)$ of $B$ has the shape




where mod $B$ is the category of finite dimensional right $B$-modules, $\mathcal{P}(B)$ is the unique postprojective component of $\Gamma(\text{mod } B)$ containing all the indecomposable projective $B$-modules, $\mathcal{Q}(B)$ is the unique preinjective component of $\Gamma(\text{mod } B)$ containing all the indecomposable injective $B$-modules, and $\mathcal{R}(B)$ is the (non-empty) regular part consisting of the remaining components of $\Gamma(\text{mod } B)$. We recall also that:

- the regular part $\mathcal{R}(B)$ of the Auslander–Reiten quiver $\Gamma(\text{mod } B)$ is a disjoint union of the $\mathbb{P}_1(K)$-family
$$\boldsymbol{\mathcal{T}}^B = \{\mathcal{T}_\lambda^B\}_{\lambda \in \mathbb{P}_1(K)}$$
of pairwise orthogonal standard stable tubes $\mathcal{T}_\lambda^B$, where $\mathbb{P}_1(K)$ is the projective line over $K$,
- the family $\boldsymbol{\mathcal{T}}^B$ separates the postprojective component $\mathcal{P}(B)$ from the preinjective component $\mathcal{Q}(B)$,
- the module category mod $B$ is controlled by the Euler quadratic form $q_B : K_0(B) \longrightarrow \mathbb{Z}$ of the algebra $B$.

In Volume 3, we study the representation-infinite tilted algebras $B = \operatorname{End} T_{KQ}$ of a Euclidean type $Q$, where $T_{KQ}$ is a tilting $T_{KQ}$-module. We give a fairly complete description of their indecomposable modules, their module categories mod $B$, and the Auslander–Reiten quivers $\Gamma(\operatorname{mod} B)$.

The aim of the present chapter is to introduce concepts playing a fundamental rôle in the classification of arbitrary representation-infinite tilted algebras of Euclidean type, presented in Chapter XVII.

In Section 1, we introduce the concepts of a one-point extension and a one-point coextension of an algebra, and we discuss a behavior of almost split sequences under the one-point extension and the one-point coextension procedure.

In Section 2, we introduce the concepts of a tubular extension and a tubular coextension of an algebra, and the related concepts of ray tubes and coray tubes. As we shall see in Chapter XVII, the components of a representation-infinite tilted algebra of Euclidean type that are neither postprojective nor preinjective, are ray tubes or coray tubes.

In Section 3, we show that the concepts of the tubular extension and the tubular coextension of an algebra coincide with the concepts of a branch extension and a branch coextension of an algebra.

In Section 4, we discuss the structure of the module categories mod $B$ and mod $B'$ of a tubular extension $B$ and a tubular coextension $B'$ of a concealed algebra $A$ of Euclidean type, and we introduce the concept of the tubular type of such algebras. The study we start in Section 4 is continued in Chapter XVII. We show there that every representation-infinite tilted algebra of Euclidean type is either a domestic tubular extension or a domestic tubular coextension of a concealed algebra of Euclidean type.

## XV.1. One-point extensions and one-point coextensions of algebras

We start by explaining the idea of a one-point extension algebra. Assume that $B$ is a $K$-algebra such that the quiver $Q_B$ of $B$ has a source vertex 0. We form a $K$-algebra $A$ in such a way that the quiver $Q_A$ of $A$ is obtained by deleting from the quiver $Q_B$ of $B$ the source 0, as well as all the arrows passing through 0. We are interested in the relation between the representation theories of $B$ and of the algebra $A$.

Let $e_0$ be the idempotent of $B$ corresponding to the source vertex 0, and we set

$$A = (1 - e_0)B(1 - e_0).$$

Since the algebra $A$ is isomorphic the quotient algebra $B/Be_0B$ of $B$ (modulo the two-sided ideal $Be_0B$ generated by the idempotent $e_0$), then the canonical algebra epimorphism

$$B \longrightarrow A \cong B/Be_0B$$

induces an embedding of module categories $\operatorname{mod} A \hookrightarrow \operatorname{mod} B$, called the standard embedding.

On the other hand, because 0 is a source of $Q_B$, the injective right $B$-module $I(0)_B = D(Be_0)$ is simple and, consequently, there is a $K$-algebra isomorphism $e_0Be_0 \cong \operatorname{End}(D(Be_0)) \cong K$, and

$$(1-e_0)Be_0 \cong \operatorname{Hom}_B(D(Be_0), D(B(1-e_0))) = 0.$$

Moreover, the (right) $K$-vector space $X = e_0B(1-e_0)$ has a canonical right $A$-module structure and a canonical left $K$-module structure induced by the right one (that is, defined by the formula $\lambda \cdot x = x\lambda$, for all $x \in X$ and $\lambda \in K$); they define a $K$-$A$-bimodule structure ${}_KX_A$ on $X$, see (I.2.10). It follows that we can view the algebra $B$ in the matrix form

$$B = \begin{bmatrix} (1-e_0)B(1-e_0) & (1-e_0)Be_0 \\ \\ e_0B(1-e_0) & e_0Be_0 \end{bmatrix} = \begin{bmatrix} A & 0 \\ {}_KX_A & K \end{bmatrix},$$

where $A = (1-e_0)B(1-e_0) \cong B/Be_0B$, ${}_KX_A = X = e_0B(1-e_0)$, $K \cong e_0Be_0$, and the multiplication is induced from the $K$-$A$-bimodule structure of ${}_KX_A$, see (A.2.7) of Volume 1. Because the right ideal

$$e_0B = \begin{bmatrix} 0 & 0 \\ {}_KX_A & K \end{bmatrix} \quad \text{of} \quad B = \begin{bmatrix} A & 0 \\ {}_KX_A & K \end{bmatrix}$$

is an indecomposable projective $B$-module, then the $A$-module ${}_KX_A$ identified with the $B$-submodule $\begin{bmatrix} 0 & 0 \\ {}_KX_A & 0 \end{bmatrix}$ of $e_0B$ equals the radical $\operatorname{rad} e_0B$ of $e_0B$, that is, we make the identification

$$\operatorname{rad} e_0B = \begin{bmatrix} 0 & 0 \\ {}_KX_A & 0 \end{bmatrix} \equiv {}_KX_A.$$

These considerations, already used implicitly, for instance in (VII.2.5) and (IX.4), and their duals, lead to the following definitions.

**1.1. Definition.** Let $A$ be a $K$-algebra, and $X$ be a right $A$-module.

(a) The **one-point extension** of $A$ by $X$, which we denote by $A[X]$, is the $2 \times 2$-matrix algebra

$$A[X] = \begin{bmatrix} A & 0 \\ {}_K X_A & K \end{bmatrix}$$

with the ordinary addition of matrices, and the multiplication induced from the usual $K$-$A$-bimodule structure ${}_K X_A$ of $X$, see (A.2.7).

(b) The **one-point coextension** of $A$ by $X$, which we denote by $[X]A$, is the $2 \times 2$-matrix algebra

$$[X]A = \begin{bmatrix} K & 0 \\ DX & A \end{bmatrix}$$

with the ordinary addition of matrices, and the multiplication induced from the $A$-$K$-bimodule structure of $DX = \mathrm{Hom}_K({}_K X_A, K)$ induced by the $K$-$A$-bimodule structure of ${}_K X_A$, see Section I.2.9 of Volume 1.

We recall that given two $K$-algebras $A$, $C$, and a finite dimensional $C$-$A$-bimodule ${}_C X_A$, the set

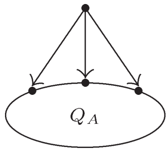
$$B = \begin{bmatrix} A & 0 \\ {}_C X_A & C \end{bmatrix}$$

of all matrices $\left[\begin{smallmatrix} a & 0 \\ x & c \end{smallmatrix}\right]$, where $a \in A$, $c \in C$, and $x \in X$, endowed with the usual matrix addition and the multiplication given by the formula

$$\begin{bmatrix} a & 0 \\ x & c \end{bmatrix} \cdot \begin{bmatrix} a' & 0 \\ x' & c' \end{bmatrix} = \begin{bmatrix} aa' & 0 \\ xa'+cx' & cc' \end{bmatrix},$$

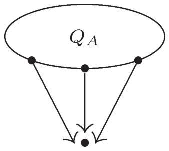
is a finite dimensional $K$-algebra with identity element $1 = e_A + e_C$, where $e_A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $e_C = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, see (A.2.7).

It is easy to see that the quiver $Q_{A[X]}$ of the one-point extension algebra $A[X]$ contains the quiver $Q_A$ of $A$ as a full convex subquiver, and there is a single additional point in $Q_{A[X]}$, which is a source vertex. One may thus visualise the quiver $Q_{A[X]}$ of $A[X]$ as follows



Conversely, the considerations preceding the definition show that any algebra $B$ having a source in the quiver $Q_B$ can be written as a one-point extension $A[X]$ of a quotient $A$ of $B$ by the two-sided ideal generated by the idempotent corresponding to this source.

Dually, the quiver $Q_{[X]A}$ of the one-point coextension $K$-algebra $[X]A$ contains the quiver $Q_A$ of $A$, as a full convex subquiver, and there is a single additional point in $Q_{[X]A}$, which is a sink vertex. One may thus visualise the quiver $Q_{[X]A}$ of $[X]A$ as follows




In this chapter, and contrary to our custom in this book, but for the sake of brevity, we only state the results for one-point extensions, but not their duals (for one-point coextensions). We urge the reader to do the primal-dual translation work.

For our purposes, an equivalent description of the category mod $A[X]$ in terms of the representations of bimodules is needed. It is well-known that modules over a $2 \times 2$ triangular matrix algebra may be represented as triples, each consisting of a pair of modules and a homomorphism.

Now we illustrate the definition with two simple examples.

• Assume that $A = K$ and $X = K$. Then the one-point extension $A[X]$ of the algebra $K$ by $X = K$ is the algebra

$$K[K] = \begin{bmatrix} K & 0 \\ K & K \end{bmatrix}$$

consisting of $2 \times 2$ lower triangular matrices with coefficients in $K$. In other words, the one-point extension $K[K]$ is the path $K$-algebra of the Dynkin quiver $1 \circ \longleftarrow \circ 2$.

• Assume that $A = K$ and $X = K^2$. Then the one-point extension $A[X]$ of the algebra $K$ by $X = K^2$ is the Kronecker algebra

$$K[K^2] = \begin{bmatrix} K & 0 \\ K^2 & K \end{bmatrix},$$

see (I.2.5). Equivalently, $K[K^2]$ is the path $K$-algebra of the Kronecker quiver $1 \circ \leftleftarrows \circ 2$, of the Euclidean type $\widetilde{\mathbb{A}}_1$.

Given a $K$-algebra $A$ and a $K$-$A$-bimodule ${}_KX_A$, we define the $K$-category

$$\operatorname{rep}(X) = \operatorname{rep}({}_KX_A) \tag{1.2}$$

of all $K$-linear **representations of the bimodule** ${}_KX_A$ as follows, see (A.2.7) in Volume 1.

(i) An object $M = (M_0, M_1, \psi_M)$ in $\operatorname{rep}({}_KX_A)$ consists of a $K$-vector space $M_0$, a right $A$-module $M_1$, and a homomorphism $\psi_M : M_0 \otimes_K X_A \longrightarrow M_1$ of right $A$-modules.

(ii) A morphism from $M = (M_0, M_1, \psi_M)$ to $M' = (M'_0, M'_1, \psi_{M'})$ in $\operatorname{rep}({}_KX_A)$ is a pair $f = (f_0, f_1)$, where $f_0 : M_0 \longrightarrow M'_0$ is a homomorphism of $K$-vector spaces and $f_1 : M_1 \longrightarrow M'_1$ is a homomorphism of $A$-modules, which are compatible with the structural homomorphisms $\psi_M$ and $\psi_{M'}$, that is, the following square commutes

$$\begin{array}{ccc} M_0 \otimes_K X_A & \xrightarrow{\psi_M} & M_1 \\ {\scriptstyle f_0\otimes 1_X}\downarrow & & \downarrow{\scriptstyle f_1} \\ M'_0 \otimes_K X_A & \xrightarrow{\psi_{M'}} & M'_1. \end{array}$$

(iii) The composition of morphisms in $\operatorname{rep}({}_KX_A)$ is induced by the composition of homomorphisms in $\operatorname{mod} K$ and $\operatorname{mod} A$, respectively.

(iv) The direct sum of two objects

$$M = (M_0, M_1, \psi_M) \quad \text{and} \quad M' = (M'_0, M'_1, \psi_{M'})$$

in $\operatorname{rep}({}_KX_A)$ is the object in $\operatorname{rep}({}_KX_A)$

$$M \oplus M' = (M_0 \oplus M'_0, M_1 \oplus M'_1, \psi_M \oplus \psi'_M)$$

in $\operatorname{rep}({}_KX_A)$.

It is easy to check that $\operatorname{rep}({}_KX_A)$ is an additive $K$-category. Moreover, using the **adjunction isomorphism** (I.2.11)

$$\operatorname{Hom}_A(M_0 \otimes_K X, M_1) \cong \operatorname{Hom}_K(M_0, \operatorname{Hom}_A(X, M_1)),$$

we see that the category $\operatorname{rep}({}_KX_A)$ is equivalent to the category $\overline{\operatorname{rep}}({}_KX_A)$ defined as follows.

(i) An object $M = (M_0, M_1, \varphi_M)$ in $\overline{\operatorname{rep}}({}_KX_A)$ consists of a $K$-vector space $M_0$, a right $A$-module $M_1$ and a homomorphism of $K$-vector spaces $\varphi_M : M_0 \longrightarrow \operatorname{Hom}_A({}_KX_A, M_1)$.

(ii) A morphism from $M = (M_0, M_1, \varphi_M)$ to $M' = (M'_0, M'_1, \varphi_{M'})$ in $\overline{\mathrm{rep}}({}_K X_A)$ is a pair $f = (f_0, f_1)$, where $f_0 : M_0 \longrightarrow M'_0$ is a homomorphism of $K$-vector spaces, and $f_1 : M_1 \longrightarrow M'_1$ is a homomorphism of $A$-modules, which are compatible with the structural homomorphisms $\varphi_M$ and $\varphi_{M'}$, that is, the following square commutes

$$\begin{array}{ccc} M_0 & \xrightarrow{\varphi_M} & \mathrm{Hom}_A({}_K X_A, M_1) \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_A({}_K X_A, f_1)} \\ M'_0 & \xrightarrow{\varphi_{M'}} & \mathrm{Hom}_A({}_K X_A, M'_1). \end{array}$$

(iii) The composition of morphisms in $\overline{\mathrm{rep}}({}_K X_A)$ is induced from the composition of homomorphisms in $\mathrm{mod}\, K$ and $\mathrm{mod}\, A$, respectively. The direct sum is defined componentwise.

It is easy to check that $\overline{\mathrm{rep}}({}_K X_A)$ is an additive $K$-category. A $K$-linear category equivalence

$$H : \mathrm{rep}({}_K X_A) \xrightarrow{\simeq} \overline{\mathrm{rep}}({}_K X_A)$$

is defined by assigning to any object $M = (M_0, M_1, \psi_M)$ of $\mathrm{rep}({}_K X_A)$ the object $H(M) = (M_0, M_1, \overline{\psi_M})$ of $\overline{\mathrm{rep}}({}_K X_A)$, where

$$\overline{\psi_M} : M_0 \longrightarrow \mathrm{Hom}_A({}_K X_A, M_1)$$

is the $K$-linear map adjoint to the homomorphism $\psi_M : M_0 \otimes_K X_A \longrightarrow M_1$ of right $A$-modules, that is, $\overline{\psi_M}(m_0)(x) = \psi_M(m_0 \otimes x)$, for all $m_0 \in M_0$ and $x \in X$. We also set $H(f_0, f_1) = (f_0, f_1)$.

**1.3. Lemma.** *Under the notation introduced above, the following two statements hold.*

(a) *The additive $K$-categories $\mathrm{rep}(X)$ and $\overline{\mathrm{rep}}(X)$ are abelian, and there exist $K$-linear equivalences of categories*

$$\mathrm{mod}\, A[X] \xrightarrow[\simeq]{F} \mathrm{rep}({}_K X_A) \xrightarrow[\simeq]{H} \overline{\mathrm{rep}}({}_K X_A).$$

(b) *If $M$ is a module in $\mathrm{mod}\, A[X]$ and $F(M) = (M_0, M_1, \psi_M)$, then the dimension vector $\mathbf{dim}\, M$ of $M$ has the form*

$$\mathbf{dim}\, M = (\mathbf{dim}\, M_1, \mathbf{dim}\, M_0),$$

*where $\mathbf{dim}\, M_0 = \dim_K M_0$ and $\mathbf{dim}\, M_1$ is the dimension vector of the $A$-module $M_1$, with $(\mathbf{dim}\, M)_a = (\mathbf{dim}\, M_1)_a = \dim_K M e_a$, for any vertex $a \neq 0$ of the quiver $Q_{A[X]}$ of $A[X]$. Here $0 \in (Q_{A[X]})_0$ is the source vertex defined by the one-point extension structure of the algebra $A[X]$.*

**Proof.** (a) The second equivalence $H$ is described above. To establish the first one, we define a $K$-linear functor

$$F : \operatorname{mod} A[X] \longrightarrow \operatorname{rep}({}_K X_A)$$

as follows. Let, as before, 0 denote the source which belongs to the quiver $Q_{A[X]}$ of the one-point extension algebra $A[X]$, but not to the quiver $Q_A$ of $A$, and $e_0 \in A[X]$ denote the corresponding idempotent.

For a right $A[X]$-module $M$, we set $M_0 = Me_0$ and $M_1 = M(1-e_0)$, and we denote by $\psi_M : M_0 \otimes_K X_A \longrightarrow M_1$ the homomorphism induced by the multiplication map $me_0 \otimes x \mapsto me_0 x$ for $m \in M$ and $x \in X$, where we use that $X = e_0 A[X](1-e_0)$. It is easy to see that $F(M) = (M_0, M_1, \psi_M)$ is an object of the category $\operatorname{rep}({}_K X_A)$. If $f : M \longrightarrow M'$ is a homomorphism of $A[X]$-modules, we define $f_0 : M_0 \longrightarrow M'_0$ and $f_1 : M_1 \longrightarrow M'_1$ to be the restrictions of $f$ to $M_0$ and $M_1$, respectively. This is possible, because, for any $m \in M$, we have $f(me_0) = f(m)e_0 \in M'_0$ and $f(m(1-e_0)) = f(m)(1-e_0) \in M'_1$. It is clear that $F(f) = (f_0, f_1)$ is a morphism from $F(M)$ to $F(M')$ in the category $\operatorname{rep}({}_K X_A)$. A routine calculation shows that we have defined an additive $K$-linear functor $F : \operatorname{mod} A[X] \longrightarrow \operatorname{rep}({}_K X_A)$.

We also define a functor

$$G : \operatorname{rep}({}_K X_A) \longrightarrow \operatorname{mod} A[X]$$

as follows. For an object $M = (M_0, M_1, \psi_M)$ in $\operatorname{rep}({}_K X_A)$, we let $G(M_0, M_1, \psi_M)$ be the right $A[X]$-module having

$$G(M) = M_1 \oplus M_0$$

as underlying vector space, with the multiplication

$$\cdot : G(M) \times A[X] \longrightarrow G(M)$$

defined by the formula

$$(m_1, m_0) \cdot \begin{bmatrix} a & 0 \\ x & \lambda \end{bmatrix} = (m_1 a + \psi_M(m_0 \otimes x), m_0 \lambda),$$

for $(m_1,m_0) \in G(M)$ and $\left[\begin{smallmatrix} a & 0 \\ x & \lambda \end{smallmatrix}\right] \in A[X]$. If

$$(f_0, f_1) : (M_0, M_1, \psi_M) \longrightarrow (M'_0, M'_1, \psi_{M'})$$

is a morphism in the category $\operatorname{rep}({}_K X_A)$, we define the $K$-linear map

$$G(f_0, f_1) : G(M_0, M_1, \psi_M) \longrightarrow G(M'_0, M'_1, \psi_{M'})$$

by the formula

$$G(f_0, f_1)(m_1, m_0) = (f_1(m_1), f_0(m_0)).$$

Then $G(f_0, f_1)$ is indeed a homomorphism of $A[X]$-modules, because the following equalities hold

$$\begin{aligned} G(f_0, f_1)\Big((m_1, m_0) \cdot \left[\begin{smallmatrix} a & 0 \\ x & \lambda \end{smallmatrix}\right]\Big) &= G(f_0, f_1)(m_1 a + \psi_M(m_0 \otimes x), m_0\lambda) \\ &= (f_1(m_1 a) + f_1\psi_M(m_0 \otimes x), f_0(m_0\lambda)) \\ &= (f_1(m_1)a + \psi_{M'}(f_0 \otimes 1_X)(m_0 \otimes x), f_0(m_0)\lambda) \\ &= (f_1(m_1)a + \psi_{M'}(f_0(m_0) \otimes x), f_0(m_0)\lambda) \\ &= (f_1(m_1), f_0(m_0)) \cdot \left[\begin{smallmatrix} a & 0 \\ x & \lambda \end{smallmatrix}\right] \\ &= [G(f_0, f_1)(m_1, m_0)] \cdot \left[\begin{smallmatrix} a & 0 \\ x & \lambda \end{smallmatrix}\right], \end{aligned}$$

for all $(m_1,m_0) \in G(M)$ and $\left[\begin{smallmatrix} a & 0 \\ x & \lambda \end{smallmatrix}\right] \in A[X]$.

It is easily shown that $F$ and $G$ are additive $K$-linear functors, and quasi-inverse to each other.

(b) The description of the functors $F$ and $G$ shows that, if a triple $M = (M_0, M_1, \varphi_M)$ is viewed as a right $A[X]$-module via $G$, then the dimension vector $\mathbf{dim}\, M$ of $M$ in $\mathrm{mod}\, A[X]$ is computed as follows. If $a$ is a point in the quiver $Q_{A[X]}$ of $A[X]$, then $(\mathbf{dim}\, M)_a = (\mathbf{dim}\, M_1)_a = \dim_K M e_a$, if $a \neq 0$, and $(\mathbf{dim}\, M)_0 = \dim_K M_0$. This finishes the proof. □

The reader is referred to (I.2.4) and (I.2.5) for simple examples explaining the functors $F$ and $H$ of Lemma (1.3). In the sequel, the equivalences $F$ and $H$ of (1.3) are treated as identifications.

The category $\overline{\mathrm{rep}}({}_K X_A)$ being more suited for our purposes, we consider in fact $A[X]$-modules as being objects in $\overline{\mathrm{rep}}({}_K X_A)$.

Another consequence of (1.3) is the following useful fact.

**1.4. Corollary.** *Let $A[X]$ be a one-point extension algebra as above. Then there exist two essentially distinct full and faithful embeddings of* $\mathrm{mod}\, A$ *inside* $\mathrm{mod}\, A[X]$ *preserving the indecomposablity:*

(a) *the standard embedding of* $\mathrm{mod}\, A$ *inside* $\mathrm{mod}\, A[X]$*, that associates to an $A$-module $M$ the triple* $(0, M, 0)$ (*we simply identify* $(0, M, 0)$ *with $M$*),

(b) *the functor associating to an $A$-module $M$ the triple*

$$\overline{M} = (\mathrm{Hom}_A(X, M), M, 1_{\mathrm{Hom}_A(X,M)}).$$

□

We now use the embeddings of (1.4) to describe those almost split sequences in the category $\operatorname{mod} A[X]$ whose right end term is an $A$-module. We need a technical lemma.

**1.5. Lemma.** *Let $A$ be an algebra, and $X$ be an $A$-module.*

(a) *If $f : L \longrightarrow M$ is left minimal almost split in* $\operatorname{mod} A$*, then*

$$(1, f) : \overline{L} \longrightarrow (\operatorname{Hom}_A(X, L), M, \operatorname{Hom}_A(X, f))$$

*is left minimal almost split in* $\operatorname{mod} A[X]$.

(b) *If $g : M \longrightarrow N$ is right minimal almost split in* $\operatorname{mod} A$*, and*

$$j : \operatorname{Ker} \operatorname{Hom}_A(X, g) \longrightarrow \operatorname{Hom}_A(X, M)$$

*denotes the inclusion, then*

$$(0, g) : (\operatorname{Ker} \operatorname{Hom}_A(X, g), M, j) \longrightarrow N$$

*is right minimal almost split in* $\operatorname{mod} A[X]$.

**Proof.** (a) Clearly, $(1, f)$ is a morphism

$$\overline{L} = (\operatorname{Hom}_A(X, L), L, 1) \longrightarrow (\operatorname{Hom}_A(X, L), M, \operatorname{Hom}_A(X, f)).$$

If it were a section, then so would be $f$, a contradiction. Thus $(1, f)$ is not a section.

Let $u = (u_0, u_1) : \overline{L} \longrightarrow (U_0, U_1, \varphi_U) = U$ be a morphism which is not a section. We claim that $u_1 : L \longrightarrow U_1$ is not a section in $\operatorname{mod} A$. For, assume to the contrary that this is the case. Then there exists $u_1' : U_1 \longrightarrow L$ such that $u_1' \circ u_1 = 1_L$. Hence the pair of maps $u' = (\operatorname{Hom}_A(X, u_1') \circ \varphi_U, u_1')$ is a morphism from $(U_0, U_1, \varphi_U)$ to $\overline{L}$, which satisfies $u' \circ u = 1_{\overline{L}}$, because

$$\operatorname{Hom}_A(X, u_1') \circ \varphi_U \circ u_0 = \operatorname{Hom}_A(X, u_1') \circ \operatorname{Hom}_A(X, u_1) = 1,$$

contrary to our hypothesis that $u$ is not a section. This establishes our claim.

Because, by hypothesis, $f$ is left almost split, there exists a homomorphism $\widetilde{u}_1 : M \longrightarrow U_1$ in $\operatorname{mod} A$ such that $\widetilde{u}_1 \circ f = u_1$. Then the pair of homomorphisms $\widetilde{u} = (u_0, \widetilde{u}_1)$ is a morphism

$$(\operatorname{Hom}_A(X, L), M, \operatorname{Hom}_A(X, f)) \longrightarrow (U_0, U_1, \varphi_U).$$

Indeed, we have

$$\operatorname{Hom}_A(X, \widetilde{u}_1) \circ \operatorname{Hom}_A(X, f) = \operatorname{Hom}_A(X, u_1) = \varphi_U u_0,$$