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Simon R. Blackburn , Peter M. Neumann , Geetha Venkataraman
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To
Kay and Keith Blackburn,
Sylvia Neumann,
and Uttara Rangajaran

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Preface

This book has grown out of a series of lectures given in the Advanced Algebra Class at Oxford in Michaelmas Term 1991 and Hilary Term 1992, that is to say from October 1991 to March 1992. The focus was—and is—the big question

how many groups of order n are there?

Two of the lectures were given by Professor Graham Higman, FRS, two by Simon R. Blackburn and the rest by Peter M. Neumann. Notes were written up week by week by Simon Blackburn and Geetha Venkataraman and those notes formed the original basis of this work. They have, however, been re-worked and updated to include recent developments.

The lectures were designed for graduate students in algebra and the book has been drafted with a similar readership in mind. It presupposes undergraduate knowledge of group theory—up to and including Sylow's theorems, a little knowledge of how a group may be presented by generators and relations, a very little representation theory from the perspective of module theory and a very little cohomology theory—but most of the basics are expounded here and the book should therefore be found to be more or less self-contained. Although it remains a work principally devoted to connected exposition of an agreeable theory, it does also contain some material that has not hitherto been published, particularly in Part IV.

We owe thanks to a number of friends and colleagues: to Graham Higman for his contribution to the lectures; to members of the original audience for their interest and their comments; to Laci Pyber for comments on an early draft; to Mike Newman for permission to include unpublished work of himself and Craig Seeley; to Eira Scourfield for guidance on the literature of analytic

number theory; to Juliette White for comments on the earlier chapters of the book and for help with proofreading our first draft. We would also like to acknowledge the support of the Mathematical Sciences Foundation, St. Stephen's College, Delhi, The Indian Institute of Science, Bangalore and our respective home institutions. Geetha Venkataraman would also like to acknowledge the encouragement and support extended by Uttara, Mahesh and Shantha Rangarajan and her parents WgCdr P. S. Venkataraman and Visalakshi Venkataraman. Professor Dinesh Singh has been a mentor providing much needed support, encouragement and intellectual fellowship. We record our gratitude to an anonymous friendly referee for constructive suggestions and for drawing our attention to some recent references that we had missed. We are also very grateful to the editorial staff of Cambridge University Press for their great courtesy, enthusiasm and helpfulness.

Turning lecture notes into a book involves much hard work. Inevitably that work has fallen unequally on the three authors. The senior author, too happy to have relied on the excellent principle *juniores ad labores* (which he admits to having embraced less enthusiastically when he was younger), is glad to have the opportunity to acknowledge that all the hard work has been done by his two colleagues, whom he thanks very warmly.

SRB, IIMN, GV: 25.xi.2006