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Excerpt
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# 1 Introduction

If it were possible for geoloscientists and engineers to know the locations of oil and gas, the locations and transmissivity of faults, the porosity, the permeability, and the multi-phase flow properties such as relative permeability and capillary pressure at all locations in a reservoir, it would be conceptually possible to develop a mathematical model that could be used to predict the outcome of any action. The relationship of the *model* variables, $m$, describing the system to observable variables or *data*, $d$, is denoted

$$g(m) = d.$$

If the model variables are known, outcomes can be predicted, usually by running a numerical reservoir simulator that solves a discretized approximation to a set of partial differential equations. This is termed the *forward problem*.

Most oil and gas reservoirs are inconveniently buried beneath thousands of feet of overburden. Direct observations of the reservoir are available only at well locations that are often hundreds of meters apart. Indirect observations are typically made at the surface, either at the well-head (production rates and pressures) or at distributed locations (e.g. seismic). In the *inverse problem*, the observations are used to determine the variables that describe the system. Real observations are contaminated with errors, $\epsilon$, so the inverse problem is to "solve" the set of equations

$$d_{\text{obs}} = g(m) + \epsilon$$

for the model variables, with the goal of making accurate predictions of future performance.

## 1.1 The forward problem

In a forward problem, the physical properties of some system (system or model parameters) are known, and a deterministic method is available for calculating the response or outcome of the system to a known stimulus. The physical properties are referred to as system or model parameters. A typical forward problem is represented by a differential equation with specified initial and/or boundary conditions. A simple example of a forward problem of interest to petroleum engineers is the following steady-state problem for a one-dimensional flow in a porous medium:

$$\frac{d}{dx}\left(\frac{k(x)A}{\mu}\frac{dp(x)}{dx}\right) = 0, \tag{1.1}$$

for $0 < x < L$, and

$$\left.\frac{dp}{dx}\right|_{x=L} = -\frac{q\mu}{k(L)A}, \tag{1.2}$$

$$p(0) = p_e \tag{1.3}$$

where $A$ (cross sectional area to flow in cm$^2$), $\mu$ (viscosity in cp), $q$ (flow rate in cm$^3$/s), and pressure $p_e$ (atm) are assumed to be constant. The length of the system in cm is represented by $L$. The function $k(x)$ represents the permeability field in Darcies. This steady-state problem could describe linear flow in either a core or a reservoir. For this forward problem, the model parameters, which are assumed to be known, are $A$, $L$, $\mu$, and $k(x)$. The stimulus for the system (reservoir or core) is provided by prescribing $q$ (the flow rate out the right-hand end) and $p(0)$ (the pressure at the left-hand end), for example, by the boundary conditions, which are assumed to be known exactly. The system output or response is the pressure field, which can be determined by solving the boundary-value problem. The solution of this steady-state boundary-value problem is given by

$$p(x) = p_e - \frac{q\mu}{A}\int_0^x \frac{1}{k(\xi)}\,d\xi. \tag{1.4}$$

If the emphasis is on the relationship between the permeability field and the pressure, we might formally write the relationship between pressure, $p_i$, at a location, $x_i$, and the permeability field as $p_i = g_i(k)$. This expression indicates that the function $g_i$ specifies the relation between the permeability field and pressure at the point $x_i$.

Forward problems of interest to us can usually be represented by a differential equation or system of differential equations together with initial and/or boundary conditions. Most such forward problems are well posed, or can be made to be well posed by imposing natural physical constraints on the coefficients of the differential equation(s) and the auxiliary conditions. Here, auxiliary conditions refer to the initial and boundary conditions. A boundary-value problem, or initial boundary-value problem, is said to be *well posed* in the sense of Hadamard [7], if the following three criteria are satisfied:

(a) the problem has a solution,
(b) the solution is unique, and
(c) the solution is a continuous function of the *problem data*.

It is important to note that the *problem data* include the functions defining the initial and boundary conditions and the coefficients in the differential equation. Thus, for the boundary-value problem of Eqs. (1.1)–(1.3), the problem data refers to $p_e$, $q\mu/k(L)A$ and $k(x)$.

If $k(x)$ were zero in some part of the core, then we can not obtain steady-state flow through the core and the pressure solution of Eq. (1.4) is not defined, i.e. the boundary-value problem of Eqs. (1.1)–(1.3) does not have a solution for $q > 0$. However, if we impose the restriction that $k(x) \geq \delta > 0$ for any arbitrarily small $\delta$ then the boundary-value problem is well posed.

If a problem is not well posed, it is said to be *ill posed*. At one time, most mathematicians believed that ill-posed problems were incorrectly formulated and nonphysical. We know now that this is incorrect and that a great deal of useful information can be obtained from ill-posed problems. If this were not so, there would be little reason to study inverse problems, as almost all inverse problems are ill posed.

## 1.2 The inverse problem

In its most general form, an inverse problem refers to the determination of the plausible physical properties of the system, or information about these properties, given the observed response of the system to some stimulus. The observed response will be referred to as observed data. For example, for the steady-state problem considered above, an inverse problem could represent the problem of determining the permeability field from pressure data measured at points in the interval $[0, L]$. Note that measured or observed data is different from the problem data introduced in the definition of a well-posed problem.

In both forward and inverse problems, the physical system is characterized by a set of model parameters, where here, a model parameter is allowed to be either a function or a scalar. For the steady single-phase flow problem, the model parameters can be chosen as the inverse permeability ($m(x) = 1/k(x)$), fluid viscosity, cross sectional area $A$ and length $L$. Note, however, the model parameters could also be chosen as $(k(x)A)/\mu$ and $L$. If we were to attempt to solve Eq. (1.1) numerically, we might discretize the permeability function, and choose $k_i = k(x_i)$ for a limited number of integers $i$ as our parameters. The choice of model parameters is referred to as a parameterization of the physical system. Observable parameters refer to those that can be observed or measured, and will simply be referred to as observed data. For the above steady-state problem, forcing fluid to flow through the porous medium at the specified rate $q$ provides the stimulus and measured values of pressure at certain locations that represent observed data. Pressure can be measured only at a well location, or in the case where the system represents a core, at locations where pressure transducers are situated. Although the relation between observed data and model parameters is often referred to as the model, we will refer to this relationship as the (assumed) theoretical model, because we wish to refer to any feasible set of specific model parameters as a model. In the continuous inverse problem, the model or model parameters may represent a function or set of functions rather than simply a discrete set of parameters. For the steady-state problem of Eqs. (1.1)–(1.3), the boundary-value problem implicitly defines the theoretical model with the explicit relation between observable parameters and the model or model parameters given by Eq. (1.4).

The inverse problem is almost never well posed. In the cases of most interest to petroleum reservoir engineers and hydrogeologists, an infinite number of equally good solutions exist. For the steady-state problem, the general inverse problem represents the determination of information about model parameters (e.g. $1/k(x)$, $\mu$, $A$, and $L$) from pressure measurements. As pressure measurements are subject to noise, measured pressure data will not, in general, be exact. The assumed theoretical model may also not be exact. For the example problem considered earlier, the theoretical model assumes constant viscosity and steady-state flow. If these assumptions are invalid, then we are using an approximate theoretical model and these modeling errors should be accounted for when generating inverse solutions.

For now, we state the general inverse problem as follows: determine plausible values of model parameters given inexact (uncertain) data and an assumed theoretical model relating the observed data to the model. For problems of interest to petroleum engineers, the theoretical model always represents an approximation to the true physical relation between physical and/or geometric properties and data. Left unsaid at this point is what is meant by plausible values (solutions) of the inverse problems. A plausible solution must of course be consistent with the observed data and physical constraints (permeability and porosity can not be negative), but for problems of interest in petroleum reservoir characterization, there will normally be an infinite number of models satisfying this criterion. Do we want to choose just one estimate? If so, which one? Do we want to determine several solutions? If so, how, why, and which ones? As readers will see, we have a very definite philosophical approach to inverse problems, one that is grounded in a Bayesian viewpoint of probability and assumes that prior information on model parameters is available. This prior information could be as simple as a geologist's statement that he or she believes that permeability is 200 md plus or minus 50. To obtain a mathematically tractable inverse problem, the prior information will always be encapsulated in a prior probability density function. Our general philosophy of the inverse problem can then be stated as follows: given prior information on some model parameters, inexact measurements of some observable parameters, and an uncertain relation between the data and the model parameters, how should one modify the prior probability density function (PDF) to include the information provided by the inexact measurements? The modified PDF is referred to as the a posteriori probability density function. In a sense, the construction of the a posteriori PDF represents the solution to the inverse problem. However, in a practical sense, one wishes to construct an estimate of the model (often, the maximum a posteriori estimate) or realizations of the model by sampling the a posteriori PDF. The process of constructing a particular estimate of the model will be referred to as estimation; the process of constructing a suite of realizations will be referred to as simulation.

Here, our emphasis is on estimating and simulating permeability and porosity fields. Our approach to the application of inverse problem theory to petroleum reservoir characterization problems may be summarized as follows.

1. Postulate a prior PDF for the model parameters from analog fields, core, logs, and seismic data. We will often assume that the prior PDF is multi-variate Gaussian, in which case the means and the covariance fully define the stochastic model.
2. Formulate the a posteriori PDF conditioned to all observed data. Data could include both production data and "hard" data (direct measurements of the variables to be estimated) for the rock property fields.
3. Construct a suite of realizations of the rock property fields by sampling the a posteriori PDF.
4. Generate a reservoir performance prediction under proposed operating conditions for each realization. This step is done using a reservoir simulator.
5. Construct statistics (e.g. histogram, mean, variance) from the set of predicted outcomes for each performance variable (e.g. cumulative oil production, water–oil ratio, breakthrough time). Determine the uncertainty in predicted performance from the statistics.

In our view, steps 2 and 3 are both vital, albeit difficult, and most of our research effort has focussed either on step 3 or on issues related to computational efficiency including the development of methods to efficiently generate sensitivity coefficients. Note that if one simply generates a set of rock property fields consistent with all observed data, but the set does not characterize the true uncertainty in the rock property fields (in our language, does not represent a correct sampling of the a posteriori PDF), steps 4 and 5 can not be expected to yield a meaningful characterization of the uncertainty in predicted reservoir performance.

# 2 Examples of inverse problems

The inverse problems examples presented in this chapter illustrate the concepts of data, model, uniqueness, and sensitivity. Each of these concepts will be developed in much greater depth in subsequent chapters. The examples are all quite simple to describe and understand, but several are difficult to solve.

## 2.1 Density of the Earth

The mass, $M$, and moment of inertia, $I$, of the Earth are related to the density distribution, $\rho(r)$, (assuming mass density is only a function of radius) by the following formulas:

$$M = 4\pi \int_0^a r^2 \rho(r)\, dr, \tag{2.1}$$

$$I = \frac{8\pi}{3} \int_0^a r^4 \rho(r)\, dr, \tag{2.2}$$

where $a$ is the radius of the Earth. If the true density is known for all $r$, then it is easy to compute the mass and the moment of inertia. In reality, the mass and moment of inertia can be estimated from measurements of the precession of the axis of rotation and the gravitational constant; the density distribution must be estimated. The data vector consists of the "observed" mass and moment of inertia of the Earth:

$$d = [M \quad I]^{\mathrm{T}} \tag{2.3}$$

and the model variable, $m = \rho(r)$, is the density. (Throughout this book, the superscript T on a matrix or vector denotes its transpose.) The relationship between the model variable and the theoretical data is

$$d = \int_0^a \begin{bmatrix} 4\pi r^2 \\ \frac{8\pi}{3} r^4 \end{bmatrix} m\, dr. \tag{2.4}$$

Note that, in this example, the dimension of the model to be estimated is infinite, while the dimension of the data space is just 2. Prior information might be a lower

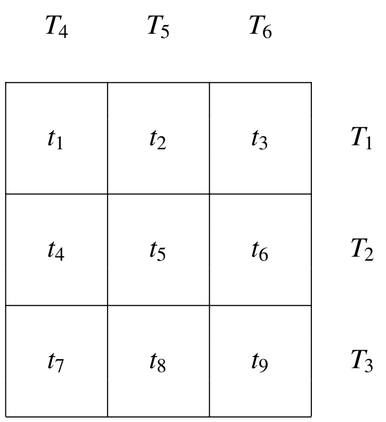


**Figure 2.1.** The array of nine blocks with traveltime parameters, $t_i$, and the six measurement locations for total traveltime, $T_i$, across the array.

bound on the density. A loose lower bound would be that density is positive. A reasonable lower bound with more information is that density is greater than or equal to 2250 kg/m$^3$. Although it is easy to generate a model that fits the data exactly, unless other information is available, the uncertainty in the estimated density at a point or a radius is unbounded.

Note also that the theoretical relationship between the density and the data in this example is only approximate as the Earth is not exactly spherical, and there is no a priori reason to believe that the density is only a function of radius.

## 2.2 Acoustic tomography

One of the simplest examples that demonstrates the concepts of sensitivity, nonuniqueness, and inconsistency is the problem of estimation of the spatial distribution of acoustic slowness (1/velocity) from measurements of traveltime along several ray paths through a solid body. For simplicity, we assume that the material properties are uniform within each of the nine blocks (Fig. 2.1) and we only consider paths that are orthogonal to the block boundaries so that refraction can be ignored and the paths remain straight. If $t$ denotes the acoustic slowness of a homogeneous block, and $T$ denotes the time required to travel a distance $D$ within or across a block, then $T = tD$. Consider a $3 \times 3$ array of blocks of various materials shown in Fig. 2.1. Each homogeneous block is 1 unit in width by 1 unit in height. Measurements of traveltime have been made for each column and each row of blocks. If the slowness of the (1, 1) block is $t_1$, the slowness of the (1, 2) block is $t_2$, and the slowness of the (1, 3) block is $t_3$, then $T_1$, the total traveltime for a sound wave to travel across the first row of blocks, is given by $T_1 = t_1 + t_2 + t_3$. Similar relations hold for the other rows and columns. If the measurements of traveltime are exact, the entire set of relations between measurements and slowness in each block is

$$\begin{aligned} T_1 &= t_1 + t_2 + t_3 \\ T_2 &= t_4 + t_5 + t_6 \\ T_3 &= t_7 + t_8 + t_9 \\ T_4 &= t_1 + t_4 + t_7 \\ T_5 &= t_2 + t_5 + t_8 \\ T_6 &= t_3 + t_6 + t_9. \end{aligned} \tag{2.5}$$

Given measured values of $T_i$, $i = 1, 2, \ldots, 6$, the inverse problem is to determine information about the acoustic slownesses, $t_j$, $j = 1, 2, \ldots, 9$. More specifically, we may wish to determine the set of all solutions of Eq. (2.6)

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \\ t_7 \\ t_8 \\ t_9 \end{bmatrix}. \tag{2.6}$$

With the notation commonly used in this book, Eq. (2.6) is written as

$$d = Gm, \tag{2.7}$$

where the data, $d$, is the vector of traveltime measurements, i.e.

$$d = [T_1 \quad T_2 \quad T_3 \quad T_4 \quad T_5 \quad T_6]^{\mathrm{T}}, \tag{2.8}$$

the model, $m$, is the vector of slowness values given by

$$m = [t_1 \quad t_2 \quad \ldots \quad t_9]^{\mathrm{T}} \tag{2.9}$$

and the sensitivity matrix, $G$, is the matrix that relates the data to the model variables and is given by

$$G = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}. \tag{2.10}$$

The reason for calling $G$ the sensitivity matrix is easily understood by examining the particular row of $G$ associated with a particular measurement. Note that there are as many rows as there are measurements. Each row has nine elements in this example, one for each model variable. The element in the $i$th row and $j$th column of $G$ gives the "sensitivity" ($\partial T_i/\partial t_j$) of the $i$th measurement to a change in the $j$th model variable. So, for example, the fourth measurement is only sensitive to $t_1$, $t_4$, and $t_7$. As can be seen easily from Eq. (2.5) or (2.6), $\partial T_4/\partial t_j = 1$ for $j = 1, 4, 7$ and $\partial T_4/\partial t_j = 0$ otherwise. Note when $\partial T_i/\partial t_j = 0$, a change in the acoustic slowness $t_j$ will not affect the value of the traveltime $T_i$, thus we can find no information on the value of $t_j$ from the measured value of $T_i$.

When we want to visualize the sensitivity for a particular measurement, we often display the row in a natural ordering, one that corresponds to the spatial distribution of model parameters; see Fig. 2.1. Here, we let $G_i$ denote the $i$th row of $G$ and display $G_2$ as:

| | | |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 1 | 1 |
| 0 | 0 | 0 |

. This display is convenient as it indicates that the second traveltime measurement only depends on the slowness values in the second row. Similarly, $G_4$ can be displayed as:

| | | |
|---|---|---|
| 1 | 0 | 0 |
| 1 | 0 | 0 |
| 1 | 0 | 0 |

, which, when compared to Fig. 2.1 shows clearly that the fourth traveltime measurement is only sensitive to the slowness values of the first column of blocks. Of course, when the models become very large, we will not display all of the numbers. Instead we will use a shading scheme that shows the strength of the sensitivity by the darkness of the grayscale.

## Solutions

Suppose that the values of acoustic slowness are such that the exact measurement of one-way traveltime in each of the columns and rows is equal to 6 units (i.e. $T_i = 6$ for all $i$). Clearly, a homogeneous model for which the slowness of each block is 2 will satisfy this data exactly, i.e. with all $t_j = 2$ and all $T_i = 6$, Eq. (2.6) is satisfied. Similarly, it is easy to see that

$$\hat{m} = [2 \quad 2 \quad 2 \quad 2+b \quad 2-b \quad 2 \quad 2-b \quad 2+b \quad 2]^{\mathrm{T}}. \tag{2.11}$$

is a solution of Eq. (2.6), for any real constant b, when all entries of the data vector are equal to 6. A little examination shows that the following models also satisfy the data exactly:

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 2 | 2 | 2 |
| 3 | 2 | 1 |

| | | |
|---|---|---|
| −2 | 0 | 8 |
| −2 | 6 | 2 |
| 10 | 0 | −4 |

| | | |
|---|---|---|
| 2+a | 2 | 2−a |
| 2 | 2 | 2 |
| 2−a | 2 | 2+a |

| | | |
|---|---|---|
| 2+b | 2−b | 2 |
| 2−b | 2+b | 2 |
| 2 | 2 | 2 |

,

**Box 1. Nonuniqueness**

The null space of $G$ is the set of all real, nine-dimensional column vectors $m$ such that $Gm = 0$. It is easy to verify that each of the following models represent vectors in the null space of $G$,

| | | |
|---|---|---|
| 0 | 1 | −1 |
| 0 | −1 | 1 |
| 0 | 0 | 0 |

| | | |
|---|---|---|
| 1 | −1 | 0 |
| −1 | 1 | 0 |
| 0 | 0 | 0 |

| | | |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | −1 |
| 0 | −1 | 1 |

| | | |
|---|---|---|
| 0 | 0 | 0 |
| 1 | −1 | 0 |
| −1 | 1 | 0 |

.

In fact, the four vectors represented by these four models represent a basis for the null space of G, so any vector in the null space of G can be written as a unique linear combination of these four vectors. If $v$ is any vector in the null space of $G$ and $m$ is a vector of acoustic slownesses that satisfies $Gm = d$ where $d$ is the vector of measured traveltimes, then the model $m + v$ also satisfies the data because

$$G(m + v) = Gm + Gv = d. \tag{2.12}$$

Thus, we can add any linear combination of models (vectors) in the null space of $G$ to a model that satisfies the traveltime data and obtain another model which also satisfies the data.

This acoustic tomography problem has an infinite number of models that satisfy the data exactly for certain data. As there are fewer traveltime data than model variables, this is not surprising. We show next, however, that for other values of the traveltime data, there are no values of acoustic slowness that satisfy Eq. (2.6).

### No solution

As measurements are always noisy, let us assume that because of the inaccuracy of the timing, the following measurements were made:

$$T_{\text{obs}} = [6.07 \quad 6.07 \quad 5.77 \quad 5.93 \quad 5.93 \quad 6.03]^{\mathrm{T}}. \tag{2.13}$$

Interestingly, despite the fact that there are fewer data than model parameters, there are *no* models that satisfy this data. Eq. (2.5) indicates that $T_1$ should be the sum of the slowness values in the first row, $T_2$ should be the sum of the slowness values in the second row, and $T_3$ should be the sum of the slowness values in the third row. Thus

$$T_1 + T_2 + T_3 = t_1 + t_2 + \cdots + t_9. \tag{2.14}$$

But $T_4$ is the sum of slowness values in column one, and similarly for $T_5$ and $T_6$ so if there are values of the model parameters that satisfy these data, we must also have

$$T_4 + T_5 + T_6 = t_1 + t_2 + \cdots + t_9. \tag{2.15}$$