1

Preliminaries

1.1 Regularly varying functions and their main properties

Regularly (and, in particular, slowly) varying functions play an important role in the subsequent exposition. In this section we will present those basic properties of the above-mentioned functions that will be used in what follows. We will often assume that the domain of the functions under consideration includes the right half-axis $(0, \infty)$ where the functions are measurable and locally integrable.

1.1.1 General properties

Definition 1.1.1. A positive (Lebesgue) measurable function $L(t)$ is said to be a *slowly varying function* (s.v.f.) as $t \to \infty$ if, for any fixed $v > 0$,

$$\frac{L(vt)}{L(t)} \to 1 \quad \text{as} \quad t \to \infty. \tag{1.1.1}$$

A function $V(t)$ is said to be a *regularly varying* (of index $-\alpha \in \mathbb{R}$) *function* (r.v.f.) as $t \to \infty$ if it can be represented as

$$V(t) = t^{-\alpha} L(t), \tag{1.1.2}$$

where $L(t)$ is an s.v.f. as $t \to \infty$.

The definition of an s.v.f. (r.v.f.) as $t \downarrow 0$ is quite similar. In what follows, the term s.v.f. (r.v.f.) will always refer, unless otherwise stipulated, to a function which is slowly (regularly) varying at infinity.

One can easily see that, similarly to (1.1.1), the convergence

$$\frac{V(vt)}{V(t)} \to v^{-\alpha} \quad \text{as} \quad t \to \infty \tag{1.1.3}$$

for any fixed $v > 0$ is a characteristic property of regularly varying functions. Thus, an s.v.f. is an r.v.f. of index zero.

Note that r.v.f.'s admit a definition that, from the first glance, appears to be more general than (1.1.3). One can define them as measurable functions such that, for all $v > 0$ from a set of positive Lebesgue measure, there exists the limit

$$\lim_{t\to\infty} \frac{V(vt)}{V(t)} =: g(v). \tag{1.1.4}$$

In this case, one necessarily has $g(v) \equiv v^{-\alpha}$ for some $\alpha \in \mathbb{R}$ and, moreover, (1.1.3) holds for all $v > 0$ (see e.g. p. 17 of [32]). The fact that the power function appears in the limit becomes natural from the obvious relation

$$g(v_1 v_2) = \lim_{t\to\infty} \frac{V(v_1 v_2 t)}{V(v_2 t)} \times \frac{V(v_2 t)}{V(t)} = g(v_1)g(v_2),$$

which is equivalent to the Cauchy functional equation for $h(u) := \ln g(e^u)$:

$$h(u_1 + u_2) = h(u_1) + h(u_2).$$

It is well known that, in 'non-pathological' cases, this equation can only have a linear solution of the form $h(u) = cu$, which means that $g(v) = v^c$.

The following functions are typical representatives of the class of s.v.f.'s: the logarithmic function and its powers $\ln^\gamma t$, $\gamma \in \mathbb{R}$, linear combinations thereof, multiple logarithms, functions with the property that $L(t) \to L = \text{const} \neq 0$ as $t \to \infty$ etc. An example of an *oscillating bounded* s.v.f. is provided by

$$L_0(t) = 2 + \sin(\ln\ln t), \qquad t > 1. \tag{1.1.5}$$

We will need the following two fundamental properties of s.v.f.'s.

Theorem 1.1.2 (Uniform convergence theorem)**.** *If $L(t)$ is an s.v.f. as $t \to \infty$, then the convergence* (1.1.1) *holds uniformly in v on any interval $[v_1, v_2]$ with $0 < v_1 < v_2 < \infty$.*

It follows from the assertion of the theorem that the uniform convergence (1.1.1) on an interval $[1/M, M]$ will also take place in the case when, as $t \to \infty$, the quantity $M = M(t)$ increases to infinity slowly enough.

Theorem 1.1.3 (Integral representation)**.** *A positive function $L(t)$ is an s.v.f. as $t \to \infty$ iff for some $t_0 > 0$ one has*

$$L(t) = c(t) \exp\left\{ \int_{t_0}^{t} \frac{\varepsilon(u)}{u}\, du \right\}, \qquad t \geqslant t_0, \tag{1.1.6}$$

where $c(t)$ and $\varepsilon(t)$ are measurable functions, with $c(t) \to c \in (0, \infty)$ and $\varepsilon(t) \to 0$ as $t \to \infty$.

For example, for the function $L(t) = \ln t$ the representation (1.1.6) holds with $c(t) = 1$, $t_0 = e$ and $\varepsilon(t) = (\ln t)^{-1}$.

Proof of Theorem 1.1.2. Put

$$h(x) := \ln L(e^x). \tag{1.1.7}$$

Then the property (1.1.1) of s.v.f.'s is equivalent to the following: for any $u \in \mathbb{R}$, one has the convergence

$$h(x+u) - h(x) \to 0 \tag{1.1.8}$$

as $x \to \infty$. To prove the theorem, we have to show that this convergence is uniform in $u \in [u_1, u_2]$ for any fixed $u_i \in \mathbb{R}$. To do this, it suffices to verify that the convergence (1.1.8) is uniform on the interval $[0,1]$. Indeed, from the obvious inequality

$$|h(x+u'+u'')-h(x)| \leqslant |h(x+u'+u'')-h(x+u')|+|h(x+u')-h(x)| \tag{1.1.9}$$

we have

$$|h(x+u) - h(x)| \leqslant (u_2 - u_1 + 1) \sup_{y\in[0,1]} |h(x+y) - h(x)|, \qquad u \in [u_1, u_2].$$

For a given $\varepsilon \in (0,1)$ and any $x > 0$ put

$$I_x := [x, x+2], \qquad I_x^* := \{u \in I_x : |h(u) - h(x)| \geqslant \varepsilon/2\},$$
$$I_{0,x}^* := \{u \in I_0 : |h(x+u) - h(x)| \geqslant \varepsilon/2\}.$$

It is clear that the sets I_x^* and $I_{0,x}^*$ are measurable and differ from each other only by a translation by x, so that $\mu(I_x^*) = \mu(I_{0,x}^*)$, where μ is the Lebesgue measure. By virtue of (1.1.8), the indicator function of the set $I_{0,x}^*$ converges to zero at any point $u \in I_0$ as $x \to \infty$. Therefore, by the dominated convergence theorem, the integral of this function, which is equal to $\mu(I_{0,x}^*)$, tends to 0, so that for large enough x_0 one has $\mu(I_x^*) < \varepsilon/2$ when $x \geqslant x_0$.

Further, for $s \in [0,1]$ the interval $I_x \cap I_{x+s} = [x+s, x+2]$ has length $2-s \geqslant 1$, so that when $x \geqslant x_0$ the set

$$(I_x \cap I_{x+s}) \setminus (I_x^* \cup I_{x+s}^*)$$

has measure $\geqslant 1 - \varepsilon > 0$ and is therefore non-empty. Let y be a point from this set. Then

$$|h(x+s) - h(x)| \leqslant |h(x+s) - h(y)| + |h(y) - h(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for $x \geqslant x_0$, which proves the required uniformity on $[0,1]$ and hence on any other fixed interval as well. The theorem is proved. □

Proof of Theorem 1.1.3. That the right-hand side of (1.1.6) is an s.v.f. is almost obvious: for any fixed positive $v \neq 1$,

$$\frac{L(vt)}{L(t)} = \frac{c(vt)}{c(t)} \exp\left\{\int_t^{vt} \frac{\varepsilon(u)}{u}\,du\right\}, \tag{1.1.10}$$

where, as $t \to \infty$, one has $c(vt)/c(t) \to c/c = 1$ and

$$\int_t^{vt} \frac{\varepsilon(u)}{u}\,du = o\left(\int_t^{vt} \frac{du}{u}\right) = o(\ln v) = o(1). \tag{1.1.11}$$

Now we prove that any s.v.f. admits the representation (1.1.6). In terms of the function (1.1.7), the required representation will be equivalent (after the change of variable $t = e^x$) to the relation

$$h(x) = d(x) + \int_{x_0}^{x} \delta(y)\,dy, \qquad (1.1.12)$$

where $d(x) = \ln c(e^x) \to d \in \mathbb{R}$ and $\delta(x) = \varepsilon(e^x) \to 0$ as $x \to \infty$, $x_0 = \ln t_0$. Therefore it suffices to establish the representation (1.1.12) for the function $h(x)$.

First of all note that $h(x)$ (like $L(t)$) is a locally bounded function. Indeed, by Theorem 1.1.2, for a large enough x_0 and all $x \geqslant x_0$

$$\sup_{0\leqslant y\leqslant 1} |h(x+y) - h(x)| < 1.$$

Hence for any $x > x_0$ we have by virtue of (1.1.9) the bound

$$|h(x) - h(x_0)| \leqslant x - x_0 + 1.$$

Further, the local boundedness and measurability of the function h mean that it is locally integrable on $[x_0, \infty)$ and therefore can be represented for $x \geqslant x_0$ as

$$h(x) = \int_{x_0}^{x_0+1} h(y)\,dy + \int_0^1 (h(x) - h(x+y))\,dy + \int_{x_0}^{x} (h(y+1) - h(y))\,dy. \qquad (1.1.13)$$

The first integral in (1.1.13) is a constant that we will denote by d. The second tends to zero as $x \to \infty$ owing to Theorem 1.1.2, so that

$$d(x) := d + \int_0^1 (h(x) - h(x+y))\,dy \to d, \qquad x \to \infty.$$

As to the third integral in (1.1.13), by the definition of an s.v.f. for its integrand one has

$$\delta(y) := h(y+1) - h(y) \to 0$$

as $y \to \infty$, which completes the proof of the representation (1.1.12). □

1.1.2 The main asymptotic properties

In this subsection we will obtain several corollaries from Theorems 1.1.2 and 1.1.3.

Theorem 1.1.4.

(i) *If L_1 and L_2 are s.v.f.'s then $L_1 + L_2$, $L_1 L_2$, L_1^b and $L(t) := L_1(at + b)$, where $a \geqslant 0$ and $b \in \mathbb{R}$, are also s.v.f.'s.*

(ii) *If L is an s.v.f. then for any $\delta > 0$ there exists a $t_\delta > 0$ such that*

$$t^{-\delta} \leqslant L(t) \leqslant t^{\delta} \quad \textit{for all} \quad t \geqslant t_\delta. \tag{1.1.14}$$

In other words, $L(t) = t^{o(1)}$ as $t \to \infty$.

(iii) *If L is an s.v.f. then for any $\delta > 0$ and $v_0 > 1$ there exists a $t_\delta > 0$ such that for all $v \geqslant v_0$ and $t \geqslant t_\delta$,*

$$v^{-\delta} \leqslant \frac{L(vt)}{L(t)} \leqslant v^{\delta}. \tag{1.1.15}$$

(iv) (Karamata's theorem) *If $\alpha > 1$ then, for the r.v.f. V in* (1.1.2)*, one has*

$$V^I(t) := \int_t^\infty V(u)\,du \sim \frac{tV(t)}{\alpha - 1} \quad \textit{as} \quad t \to \infty. \tag{1.1.16}$$

If $\alpha < 1$ then

$$V_I(t) := \int_0^t V(u)\,du \sim \frac{tV(t)}{1 - \alpha} \quad \textit{as} \quad t \to \infty. \tag{1.1.17}$$

If $\alpha = 1$ then one has the equalities

$$V_I(t) = tV(t)L_1(t) \tag{1.1.18}$$

and

$$V^I(t) = tV(t)L_2(t) \quad \textit{if} \quad \int_0^\infty V(u)du < \infty, \tag{1.1.19}$$

where the $L_i(t) \to \infty$ as $t \to \infty$, $i = 1, 2$, are s.v.f.'s.

(v) *For an r.v.f. V of index $-\alpha < 0$ put*

$$\sigma(t) := V^{(-1)}(1/t) = \inf\{u : V(u) < 1/t\}.$$

Then $\sigma(t)$ is an r.v.f. of index $1/\alpha$:

$$\sigma(t) = t^{1/\alpha} L_1(t), \tag{1.1.20}$$

where L_1 is an s.v.f. If the function L has the property

$$L\big(tL^{1/\alpha}(t)\big) \sim L(t) \tag{1.1.21}$$

as $t \to \infty$ then

$$L_1(t) \sim L^{1/\alpha}\big(t^{1/\alpha}\big). \tag{1.1.22}$$

Similar assertions hold for functions that are slowly or regularly varying as t decreases to zero.

Note that from Theorem 1.1.2 and the inequality (1.1.15) we also obtain the following property of s.v.f.'s: *for any* $\delta > 0$ *there exists a* $t_\delta > 0$ *such that for all* t *and* v *satisfying the inequalities* $t \geqslant t_\delta$, $vt \geqslant t_\delta$ *one has*

$$(1-\delta)\min\{v^\delta, v^{-\delta}\} \leqslant \frac{L(vt)}{L(t)} \leqslant (1+\delta)\max\{v^\delta, v^{-\delta}\}. \qquad (1.1.23)$$

Proof. Assertion (i) is evident (just observe that, to prove the last part of (i), one needs Theorem 1.1.2).

(ii) This property follows immediately from the representation (1.1.6) and the bound

$$\left|\int_{t_0}^{t} \frac{\varepsilon(u)}{u}du\right| = \left|\int_{t_0}^{\ln t} + \int_{\ln t}^{t}\right| = O\left(\int_{t_0}^{\ln t}\frac{du}{u}\right) + o\left(\int_{\ln t}^{t}\frac{du}{u}\right) = o(\ln t)$$

as $t \to \infty$.

(iii) To prove this property, we notice that, in relation to the expression on the right-hand side of (1.1.10), for any fixed $\delta > 0$ and $v_0 > 1$ and all sufficiently large t one has

$$v^{-\delta/2} \leqslant v_0^{-\delta/2} \leqslant \frac{c(vt)}{c(t)} \leqslant v_0^{\delta/2} \leqslant v^{\delta/2}, \qquad v \geqslant v_0,$$

and

$$\left|\int_{t}^{vt} \frac{\varepsilon(u)}{u}du\right| \leqslant \frac{\delta}{2}\ln v$$

(by virtue of (1.1.11)). From this (1.1.15) follows.

(iv) Owing to the uniform convergence theorem, one can choose an $M = M(t) \to \infty$ as $t \to \infty$ such that the convergence in (1.1.1) will be uniform in $v \in [1, M]$. Changing variables by putting $u = vt$, we obtain

$$V^I(t) = t^{-\alpha+1}L(t)\int_1^\infty v^{-\alpha}\frac{L(vt)}{L(t)}\,dv = t^{-\alpha+1}L(t)\left(\int_1^M + \int_M^\infty\right). \qquad (1.1.24)$$

If $\alpha > 1$ then, as $t \to \infty$,

$$\int_1^M \sim \int_1^M v^{-\alpha}dv \to \frac{1}{\alpha-1},$$

whereas due to property (iii) one has, for $\delta = (\alpha-1)/2$, the relation

$$\int_M^\infty < \int_M^\infty v^{-\alpha+\delta}dv = \int_M^\infty v^{-(\alpha+1)/2}dv \to 0.$$

Together these two relations mean that

$$V^I(t) \sim \frac{t^{-\alpha+1}}{\alpha-1}\, L(t) = \frac{tV(t)}{\alpha-1}.$$

The case $\alpha < 1$ is considered quite similarly, but taking into account that the convergence in (1.1.1) is uniform in $v \in [1/M, 1]$ and also the equality

$$\int_0^1 v^{-\alpha} dv = \frac{1}{1-\alpha}.$$

If $\alpha = 1$ then the first integral on the right-hand side of (1.1.24) is

$$\int_1^M \sim \int_1^M v^{-1} dv = \ln M,$$

so that if

$$\int_0^\infty V(u)\,du < \infty \tag{1.1.25}$$

then

$$V^I(t) \geqslant (1+o(1))L(t)\ln M \gg L(t) \tag{1.1.26}$$

and therefore

$$L_2(t) := \frac{V^I(t)}{tV(t)} = \frac{V^I(t)}{L(t)} \to \infty \quad \text{as} \quad t \to \infty.$$

Now note that, by virtue of property (i), L_2 is an s.v.f. if the function $V^I(t)$ is such. But, for $v > 1$,

$$V^I(t) = V^I(vt) + \int_t^{vt} V(u)\,du,$$

where the integral clearly does not exceed $(v-1)L(t)(1+o(1))$. Owing to (1.1.26) this implies that $V^I(vt)/V^I(t) \to 1$ as $t \to \infty$, which completes the proof of (1.1.19).

That (1.1.18) is true in the subcase when (1.1.25) holds is almost obvious, since

$$V_I(t) = tV(t)L_1(t) = L(t)L_1(t) = \int_0^t V(u)\,du \to \int_0^\infty V(u)\,du,$$

so that, firstly, L_1 is an s.v.f. by virtue of property (i) and, secondly, $L_1(t) \to \infty$ since $L(t) \to 0$ owing to (1.1.26).

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Now let $\alpha = 1$ and $\int_0^\infty V(u)\,du = \infty$. Then, if $M = M(t) \to \infty$ sufficiently slowly, one obtains by the uniform convergence theorem a result similar to (1.1.26) (see also (1.1.24)):

$$V_I(t) = \int_0^1 v^{-1}L(vt)\,dv \geqslant \int_{1/M}^1 v^{-1}L(vt)\,dv \sim L(t)\ln M \gg L(t).$$

Therefore $L_1(t) := V_I(t)/L(t) \to \infty$ as $t \to \infty$. Further, also by an argument similar to the previous exposition, for $v \in (0,1)$ one has

$$V_I(t) = V_I(vt) + \int_{vt}^t V(u)\,du,$$

where the last integral does not exceed $(1-v)L(t)(1+o(1)) \ll V_I(t)$, so that $V_I(t)$ (and also, by property (i), $L_1(t)$) is an s.v.f. This completes the proof of property (iv).

(v) Clearly, by the uniform convergence theorem the quantity $\sigma = \sigma(t)$ is a solution to the 'asymptotic equation'

$$V(\sigma) \sim \frac{1}{t} \quad \text{as} \quad t \to \infty \tag{1.1.27}$$

(where the symbol $\sim$ can be replaced by the equality sign provided that the function V is continuous and monotonically decreasing). Representing σ in the form $\sigma = t^{1/\alpha}L_1$, $L_1 = L_1(t)$, we obtain an equivalent relation

$$L_1^{-\alpha}L(t^{1/\alpha}L_1) \sim 1, \tag{1.1.28}$$

and it is obvious that

$$t^{1/\alpha}L_1 \to \infty \quad \text{as} \quad t \to \infty. \tag{1.1.29}$$

Fix an arbitrary $v > 0$. Substituting vt for t in (1.1.28) and for brevity putting $L_2 = L_2(t) := L_1(vt)$, we get the relation

$$L_2^{-\alpha}L(t^{1/\alpha}L_2) \sim 1, \tag{1.1.30}$$

since $L(v^{1/\alpha}t^{1/\alpha}L_2) \sim L(t^{1/\alpha}L_2)$ owing to (1.1.29) (with L_1 replaced by L_2). Now we will show by contradiction that (1.1.28)–(1.1.30) imply that $L_1 \sim L_2$ as $t \to \infty$, where the latter clearly means that L_1 is an s.v.f.

Indeed, the contrary assumption means that there exist a $v_0 > 1$ and a sequence $t_n \to \infty$ such that

$$u_n := L_2(t_n)/L_1(t_n) > v_0, \qquad n = 1, 2, \ldots \tag{1.1.31}$$

(the possible alternative case can be considered in exactly the same way). Evidently, $t_n^* := t_n^{1/\alpha}L_1(t_n) \to \infty$ by virtue of (1.1.29), so that from (1.1.28), (1.1.29) and property (iii) with $\delta = \alpha/2$ we obtain that

$$1 \sim \frac{L_2^{-\alpha}(t_n)L(t_n^{1/\alpha}L_2(t_n))}{L_1^{-\alpha}(t_n)L(t_n^{1/\alpha}L_1(t_n))} = u_n^{-\alpha}\frac{L(u_n t_n^*)}{L(t_n^*)} \leqslant u_n^{-\alpha/2} < v_0^{-\alpha/2} < 1.$$

We get a contradiction.

Note that the above argument proves the uniqueness (up to the asymptotic equivalence) of the solution to the equation (1.1.27).

Finally, the relation (1.1.22) can be proved by directly verifying (1.1.27) for $\sigma := t^{1/\alpha}L^{1/\alpha}(t^{1/\alpha})$: using (1.1.21), one has

$$V(\sigma) = \sigma^{-\alpha}L(\sigma) = \frac{L(t^{1/\alpha}L^{1/\alpha}(t^{1/\alpha}))}{tL(t^{1/\alpha})} \sim \frac{L(t^{1/\alpha})}{tL(t^{1/\alpha})} = \frac{1}{t}.$$

The desired assertion now follows owing to the above-mentioned uniqueness of the solution to the asymptotic equation (1.1.27). Theorem 1.1.4 is proved. □

1.1.3 The asymptotic properties of the transforms of r.v.f.'s (an Abelian type theorem)

For an r.v.f. $V(t)$, its Laplace transform

$$\psi(\lambda) := \int_0^\infty e^{-\lambda t}V(t)\,dt < \infty$$

is defined for any $\lambda > 0$. The following asymptotic relations hold true for the transform.

Theorem 1.1.5. *Let $V(t)$ be an r.v.f.* (i.e. it has the form (1.1.2)).

(i) *If $\alpha \in [0,1)$ then*

$$\psi(\lambda) \sim \frac{\Gamma(1-\alpha)}{\lambda}V(1/\lambda) \quad \textit{as} \quad \lambda \downarrow 0. \tag{1.1.32}$$

(ii) *If $\alpha = 1$ and $\int_0^\infty V(t)\,dt = \infty$ then*

$$\psi(\lambda) \sim V_I(1/\lambda) \quad \textit{as} \quad \lambda \downarrow 0, \tag{1.1.33}$$

where $V_I(t) = \int_0^t V(u)\,du \to \infty$ is an s.v.f. and, moreover, $V_I(t) \gg L(t)$ as $t \to \infty$.

(iii) *In any case, $\psi(\lambda) \uparrow V_I(\infty) = \int_0^\infty V(t)\,dt \leqslant \infty$ as $\lambda \downarrow 0$.*

Rewriting the relation (1.1.32), one obtains

$$V(t) \sim \frac{\psi(1/t)}{t\Gamma(1-\alpha)} \quad \text{as} \quad t \to \infty.$$

Relations of this type will also hold true in the case when, instead of the regularity of the function V, we require its monotonicity and then assume that $\psi(\lambda)$ is an

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r.v.f. as $\lambda \downarrow 0$. Assertions of this kind are referred to as *Tauberian theorems*. In the present book, we will not be using such theorems, so we will not dwell on them here.

Proof of Theorem 1.1.5. (i) For any fixed $\varepsilon > 0$ we have

$$\psi(\lambda) = \int_0^{\varepsilon/\lambda} + \int_{\varepsilon/\lambda}^{\infty}, \tag{1.1.34}$$

where owing to (1.1.17) one has the following relation for the first integral in the case $\alpha < 1$:

$$\int_0^{\varepsilon/\lambda} e^{-\lambda t} V(t)\,dt \leqslant \int_0^{\varepsilon/\lambda} V(t)\,dt \sim \frac{\varepsilon V(\varepsilon/\lambda)}{\lambda(1-\alpha)} \quad \text{as} \quad \lambda \downarrow 0. \tag{1.1.35}$$

Making the change of variables $\lambda t = u$, one can rewrite the second integral in (1.1.34) as follows:

$$\int_{\varepsilon/\lambda}^{\infty} = \frac{V(1/\lambda)}{\lambda} \int_{\varepsilon}^{\infty} e^{-u} u^{-\alpha} \frac{L(u/\lambda)}{L(1/\lambda)}\,du = \frac{V(1/\lambda)}{\lambda} \left(\int_{\varepsilon}^{2} + \int_{2}^{\infty} \right). \tag{1.1.36}$$

Here, as $\lambda \downarrow 0$, each of the two integrals on the right-hand side converges to the respective integral of $e^{-u}u^{-\alpha}$: for the former, this follows from the uniform convergence theorem (the convergence $L(u/\lambda)/L(1/\lambda) \to 1$ holds uniformly in $u \in [\varepsilon, 2]$), whereas for the latter it is a consequence of (1.1.1) and the dominated convergence theorem (since, owing to Theorem 1.1.4(iii), for all sufficiently small λ one has $L(u/\lambda)/L(1/\lambda) < u$ for $u \geqslant 2$). Therefore

$$\int_{\varepsilon/\lambda}^{\infty} \sim \frac{V(1/\lambda)}{\lambda} \int_{\varepsilon}^{\infty} u^{-\alpha} e^{-u} du. \tag{1.1.37}$$

Now observe that, as $\lambda \downarrow 0$,

$$\frac{\varepsilon V(\varepsilon/\lambda)}{\lambda} \Big/ \frac{V(1/\lambda)}{\lambda} = \varepsilon^{1-\alpha} \frac{L(\varepsilon/\lambda)}{L(1/\lambda)} \to \varepsilon^{1-\alpha}.$$

Since $\varepsilon > 0$ can be chosen arbitrary small, this relation together with (1.1.35) and (1.1.37) completes the proof of (1.1.32).

(ii) Integrating by parts and again making the change of variables $\lambda t = u$, we