Monopoles and Three-Manifolds

Originating with Andreas Floer in the 1980s, Floer homology has proved to be an effective tool in tackling many important problems in 3- and 4-dimensional geometry and topology. This book provides a comprehensive treatment of Floer homology, based on the Seiberg–Witten monopole equations. After first providing an overview of the results, the authors develop the analytic properties of the Seiberg–Witten monopole equations, assuming only a basic grounding in differential geometry and analysis. The Floer groups of a general 3-manifold are then defined, and their properties studied in detail. Two final chapters are devoted to the calculation of Floer groups, and to applications of the theory in topology.

Suitable for beginning graduate students and researchers in the field, this book provides the first full discussion of a central part of the study of the topology of manifolds since the mid 1990s.

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CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521880220

First published 2007

Printed in the United Kingdom at the University Press, Cambridge

A catalogue record for this publication is available from the British Library

ISBN 978-0-521-88022-0 hardback

To Jenny and Gigliola

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Preface

Gauge theory and related areas of geometry have been an important tool for the study of 4-dimensional manifolds since the early 1980s, when Donaldson introduced ideas from Yang–Mills theory to solve long-standing problems in topology. In dimension 3, the same techniques formed the basis of Floer's construction of his "instanton homology" groups of 3-manifolds [32]. Today, Floer homology is an active area, and there are several varieties of Floer homology theory, all with closely related structures. While Floer's construction used the anti-self-dual Yang–Mills (or *instanton*) equations, the theory presented in this book is based instead on the Seiberg–Witten equations (or *monopole* equations).

We have aimed to lay a secure foundation for the study of the Seiberg–Witten equations on a general 3-manifold, and for the construction of the associated Floer groups. Our goal has been to write a book that is complete in its coverage of several aspects of the theory that are hard to find in the existing literature, providing at the same time an introduction to the techniques from analysis and geometry that are used. We have omitted some background topics that are now well treated in several good sources: in particular, the Seiberg–Witten invariants of closed 4-manifolds and related topics in gauge theory are given only a brief exposition here. The main results of this book – the formal properties of the Floer groups that we construct – can be summarized without delving too far into the techniques which lie beneath; so we present such a summary in Chapter I. The final chapter of the book touches on some further topics and describes how the theory has been applied to questions in topology.

The definition of the Floer groups that we present here is new in some aspects. We believe that our approach to the Morse homology of a manifold with circle action has not appeared before. It is described in Section 2, along with a closely related approach to Morse theory on a manifold with boundary. Our definition of the groups that we call $\widecheck{HM}(Y)$ and $\widehat{HM}(Y)$ has roots in lectures given by Donaldson in Oxford in 1993. For the case that the first Betti number of Y is zero, a similar construction is described in [22] for the case of the instanton Floer theory, and there is related material due to Frøyshov in [40]. Another approach to the Seiberg–Witten version of Floer homology is presented by Marcolli and Wang in [71].

During the course of this work, a completely different approach to Floer homology was introduced by Ozsváth and Szabó in [93]. The construction of their "Heegaard homology" of 3-manifolds is not based on gauge theory, but appears to be entirely equivalent to the Seiberg–Witten version. Ozsváth and Szabó's theory has influenced the development of this book, most particularly because of the way in which it has clarified the formal structure of Floer homology. We have sometimes tailored our account to emphasize the similarities between the two versions. Heegaard homology has spurred tremendous activity in the topological applications of Floer theory. Chapter X provides a small sample of results from this rapidly moving field.

Acknowledgements. Gauge theory is now a mature subject, and the analysis on which it rests has deep roots. Much of the material that we present is therefore not original. When a particular argument is taken directly from a unique source, we have tried to cite the source at the relevant point in the text. More often, however, pointers to the earlier literature are to be found in the remarks collected at the end of each chapter. Among the many mathematicians who have contributed to this field, we would like to acknowledge particularly our debt to Simon Donaldson, Kim Frøyshov, Peter Ozsváth, Zoltán Szabó and Cliff Taubes.

This work was supported in part by the Institute for Advanced Study and the National Science Foundation through grants DMS-9531964, DMS-9803166, DMS-0100771, DMS-011129, DMS-0244663, DMS-0206485 and DMS-0405271. The authors would also like to thank Ron Fintushel, Larry Guth, Yi-Jen Lee, Yanki Lekili, Max Lipyanski, Tim Perutz, Yann Rollin, Jake Rasmussen, Peter Ozsváth, Zoltán Szabó and Fangyun Yang for many comments and corrections.