FRAMEWORKS, TENSEGRITIES, AND SYMMETRY

This introduction to the theory of rigid structures explains how to analyze the performance of built and natural structures under loads, paying special attention to the role of geometry. The book unifies the engineering and mathematical literatures by exploring different notions of rigidity – local, global, and universal – and how they are interrelated. Important results are stated formally, but also clarified with a wide range of revealing examples. An important generalization is to tensegrities, where fixed distances are replaced with “cables” not allowed to increase in length and “struts” not allowed to decrease in length. A special feature is the analysis of symmetric tensegrities, where the symmetry of the structure is used to simplify matters and it allows the theory of group representations to be applied. Written for researchers and graduate students in structural engineering and mathematics, this work is also of interest to computer scientists and physicists.

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CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
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University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India

103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9780521879101
DOI: 10.1017/9780511843297

First published 2022

Printed in the United Kingdom by TJ Books Limited, Padstow Cornwall

A catalogue record for this publication is available from the British Library

Library of Congress Cataloging-in-Publication Data
Names: Connelly, Robert (Mathematician), author. |
Guest, S. D. (Simon D.), author.
Title: Frameworks, tensegrities, and symmetry / Robert Connelly,
Simon D. Guest.
Description: Cambridge ; New York, NY : Cambridge University Press, 2022. |
Includes bibliographical references and index.
Identifiers: LCCN 2021029104 (print) | LCCN 2021029105 (ebook) |
ISBN 9780521879101 (hardback) | ISBN 9780511843297 (epub)
Subjects: LCSH: Structural analysis (Engineering) | Rigidity (Geometry) |
Engineering mathematics. | BISAC: MATHEMATICS / Discrete Mathematics
Classification: LCC TA645 .C655 2021 (print) | LCC TA645 (ebook) |
DDC 624.1/7–dc23
LC record available at https://lccn.loc.gov/2021029104
LC ebook record available at https://lccn.loc.gov/2021029105

ISBN 978-0-521-87910-1 Hardback

To our wives, Gail and Karen

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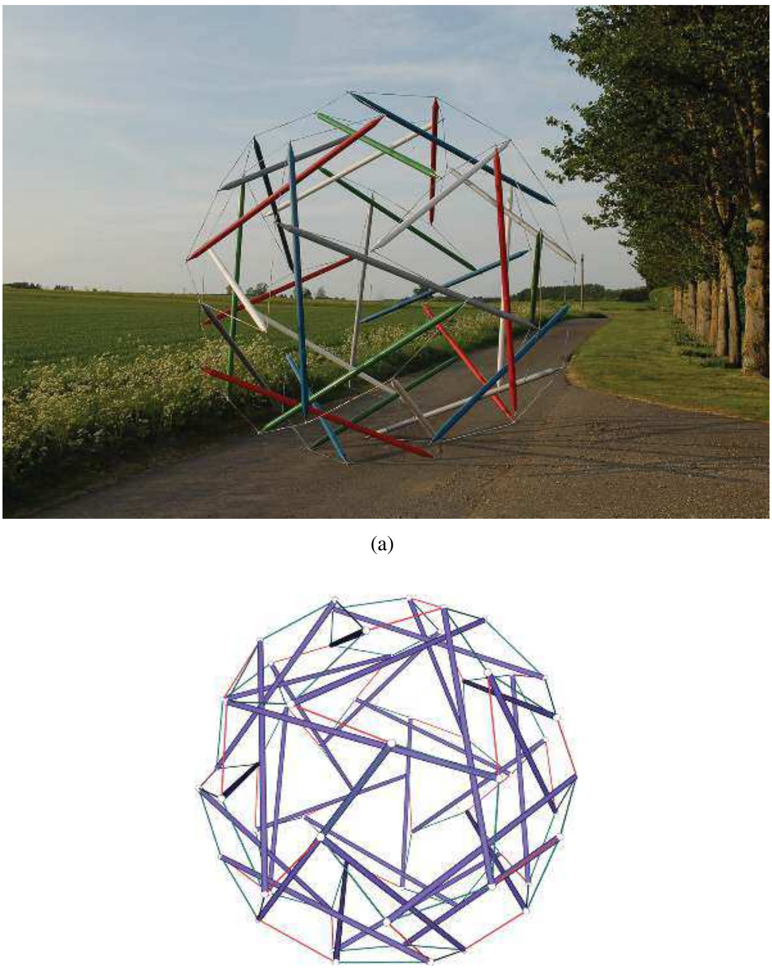
Preface

Ultimately, this book will show the reader how to generate the special geometry that allows the structure shown in Figure 0.1 to stand up. This isn't a common-or-garden structure, and understanding how it works turns out to be remarkably instructive.

This book is an attempt to build a bridge between two cultures, one the very practical and concrete, the analysis and synthesis of built structures, and the other a rigorous, but very visual, geometric understanding of the rigidity of configurations of points with distance constraints. The former is associated with an engineering point of view that dates back to Galileo, and the way it designs and analyzes structures to make sure they're safe. The latter is associated with Euclidean geometry and the rigidity of graphs, or what we call frameworks here. From about the time of the early 1970s there has been mathematical interest in the geometric rigidity of frameworks, especially with the work of Janos Baracs, Henry Crapo, and Walter Whiteley in Montreal, Canada. Janos Baracs was an engineer, trained in Hungary, who saw the connection and oneness of geometry and the problems of the rigidity of frameworks and other related structures. Our purpose here is to continue in that tradition from our own point of view as well as present some of the major advances in the theory of rigid structures.

One of us, Connelly, also became interested in the geometry of triangulated surfaces in the early 1970s, but from the point of view of trying to show that any triangulated surface in three-space (a kind of framework) is rigid. It was not the case – there is a counterexample.

Independently of these developments, in the 1940s an artist, Kenneth Snelson, created structures that seemed to magically suspend large metal bars, just with some thin cables. These were called "tensegrities" by R. Buckminster Fuller, who built up an intense following promoting the virtues of tensegrities and other related structures. The problem is, why do they stand up? What are the principles involved? In the 1970s Calladine showed how the standard engineering first-order analysis would not explain why most of Snelson's structures stood up. Of course a blunt analysis could always be applied – if bars and cables are treated like springs an analysis of the inherent potential energy will explain the stability of tensegrities. But this would be blind to the geometry of the structure, and the energy functions that would be employed were clumsy from a mathematical perspective. The simple mathematical "yes or no" question of whether the tensegrity is rigid could actually usually be answered with much simpler "energy" functions, using sums of squared lengths.



(a)

(b)

Figure 0.1 (a) A tensegrity built by David Burnett (with John Wythe) to celebrate the career of his brother Roger Burnett, who worked to elucidate the structure of the adenovirus. To help David with the design, the required geometry of the tensegrity was calculated by Ramar Pandia Raj and the authors, using the tools developed in this book in Part II (Pandia Raj and Guest, 2006). In fact the structure deviates from the ideal icosahedral form because of the effect of gravity, and this effect was later explored by Pandia Raj using the tools described in Part I (Ramar and Guest, 2011). (b) An image of the same tensegrity produced using the software described in Chapter 11.

This brings up the question of what exactly is the problem? From a mathematical point of view, the basic question is whether the configuration of the tensegrity (or bar framework) is fixed – is it rigid? Yes or no? Cables that cannot get longer, and struts that cannot get shorter, give the geometric constraints for the tensegrity. However, from the engineering/physical point of view, that is the very least you would expect to know. Perhaps a more important question is, how does the structure deform under some set of reasonable loads, when there may be some softness in the distance constraints? It seems harsh not to allow a cable not to increase in length at all, for example. Not only that, using sums of squared lengths times a constant to represent physical energies of reasonable cables and struts seems intensely unphysical, especially for struts. It turns out that these two difficulties are in fact intimately married.

This reminds us of an old joke. A mathematician and an engineer are in a hotel in separate rooms. A fire breaks out in the engineer's room, and fortunately there is a bucket of water handy. She puts out the fire with the water and goes to bed relieved. The mathematician sees all this and is quite impressed. As it happens, later, another fire breaks out in the mathematician's room, and sure enough, there is another bucket of water available as well. But he just goes to bed, secure in the knowledge that there is a solution to the "fire" problem.

A key point is that a tensegrity structure can be "self-stressed". Even without external load, each member can carry a force – tension in a cable, compression in a strut. We can associate a scalar value with this force (or the force/length), no matter what the cross-section. An engineer will need to design the member so that it can carry the force without failing, but that is not our task here. We are just interested in how the force might impact the overall rigidity – perhaps naively we assume that the members won't fail.

The approach described in this book has some unexpected and far-reaching benefits beyond buildings and bridges. For example, suppose that you have a framework with its configuration of labelled points and connecting bars, cables, or struts in three-space, say. Is there another configuration, in three-space, not just a congruent copy, that satisfies the same distance constraints? If the answer is no, then we say that the tensegrity is *globally rigid*. This is of great interest for point-location problems, or the uniqueness of protein configurations, for example. It turns out that the Snelson-type tensegrities are globally rigid. The unusual energy functions described above are just what is needed to show the global rigidity property. Indeed, there is more. The stress–energy function that is associated with the stress densities in the members determine a quadratic form that can determine the configuration, not just in the dimension that it sits, but in all higher dimensions. One consequence of this is that if the lengths of the framework are given, then the configuration can be determined by some standard quadratic programming algorithms. In case the configuration is generic (explained below), global rigidity is determined by the rank of the quadratic form. In case the quadratic form is positive semi-definite of maximal rank, then the configuration is determined in all dimensions, which we call *universal rigidity*. All this is determined modulo affine motions, which are usually easy to handle.

It also turns out that the yes-or-no question, "Is it rigid?" can be answered in certain cases to great effect when the configuration of points is generic. Technically this means that there is no non-zero polynomial relation over the integers among the coordinates of the configuration. In effect this means that if you choose the points of the configuration randomly, with probability one, your configuration will be generic. The problem is that often the most interesting structures are not generic, and it is hard to tell whether it is indeed generic. The major result is that if a configuration in the plane is generic, then there is a very efficient algorithm (now known as the pebble game) that can determine whether the bar framework is rigid or not, and it works for graphs with a large number of nodes. This has led to a large industry of determining the generic rigidity of the corresponding frameworks that is completely combinatorial, no matrices, no energies. We give a short introduction to how this combinatorial rigidity works.

Lastly, coming back to Snelson-like tensegrity structures, in Part II, we use some representation theory of finite subgroups of the orthogonal group of three-space to catalog certain classes of symmetric tensegrities, where there is one transitivity class of nodes, two transitivity classes of cables, and one transitivity class of struts.

We thank the US National Science Foundation grant numbers 0209595, 0510625, DMS-0809068, and 1564493 and the UK Engineering and Physical Sciences Research Council grant EP/D030617/1 for support during the creation of this book.