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Introduction

In Chapter 1, we introduce basic definitions and notions. We also outline some of the known algorithms devised for solving problems related to flows, cuts, and connectivities. These algorithms will be used as a basis for the discussion in subsequent chapters. The standard definitions and other topics in graph theory can be found in the book by R. Diestel [52] or other textbooks on graph theory (e.g., [10, 33]). For basic data structures for graphs, standard graph algorithms, and their complexity, see the book by R. E. Tarjan [298], for example.

1.1 Preliminaries of Graph Theory

Let $\Re$ (resp. $\Re_+$ and $\Re_-$) denote the set of reals (resp. nonnegative reals and nonpositive reals) and $\mathbf{Z}$ (resp. $\mathbf{Z}_+$ and $\mathbf{Z}_-$) denote the set of integers (resp. nonnegative integers and nonpositive integers). For a real $a \in \Re$, $\lceil a \rceil$ (resp. $\lfloor a \rfloor$) denotes the smallest integer not smaller than a (resp. the largest integer not larger than a). For two reals $a, b \in \Re$ with $a \leq b$, we denote by $[a, b]$ and (a, b) the closed interval and open intervals; i.e., the sets of reals c with $a \leq c \leq b$ and $a < c < b$, respectively.

A singleton set $\{x\}$ may be simply written as x, and "$\subset$" implies proper inclusion, whereas "$\subseteq$" means "$\subset$ or $=$". The union of a set A and a singleton set $\{x\}$ may be denoted by $A + x$.

Let V be a finite set. The cardinality of (i.e., the number of elements in) V is denoted $|V|$. Let 2^V denote the *power set* of V, i.e., the family of all subsets of V (hence $|2^V| = 2^{|V|}$). The set of all pairs of elements in a set V is denoted $\binom{V}{2}$ (hence $|\binom{V}{2}| = \binom{|V|}{2}$). We say that a subset $X \subseteq V$ *divides* another subset $Y \subseteq V$ if $X \cap Y \neq \emptyset \neq Y - X$. For two subsets $A, B \subset V$, we say that a subset $X \subseteq V$ *separates* A and B if $A \subseteq X \subseteq V - B$ or $B \subseteq X \subseteq V - A$. For two subsets $X, Y \subseteq V$, we say that X and Y *intersect* each other if $X \cap Y \neq \emptyset$, $X - Y \neq \emptyset$, and $Y - X \neq \emptyset$ hold, and we say that X and Y *cross* each other if, in addition, $V - (X \cup Y) \neq \emptyset$ holds. For a weight function $a : V \to \Re$, we denote $\sum_{v \in X} a(v)$ by $a(X)$ for all $X \subseteq V$. A set of subsets of V, $\{V_1, V_2, \ldots, V_k\}$ with

$V_i \subseteq V (i = 1, 2, \ldots, k)$, is a *partition* of V if $\bigcup_{i=1}^{k} V_i = V$ and $V_i \cap V_j = \emptyset$ holds for all $i \neq j$.

An *undirected graph* (or a *graph*) G and a *directed graph* (or a *digraph*) G are defined by a pair composed of a vertex set V and an edge set $E \subseteq V \times V$, depending on whether edges are undirected and directed, respectively, and are denoted by $G = (V, E)$. The vertex set and edge set of a graph G may be denoted by $V(G)$ and $E(G)$, respectively. We use the notation $n = |V|$ and $m = |E|$ throughout this book.

An undirected edge e with end vertices u and v is denoted by $\{u, v\}$. We say that e is *incident* with u and v, u and v are the end vertices of e, and u (resp. v) is *adjacent* to v (resp. u). A directed edge e with tail u and head v is denoted by (u, v), and the *head* (resp. *tail*) of e is denoted by $h(e)$ (resp. $t(e)$). In this case, we say that $e = (u, v)$ is incident from u to v. An edge with the same end vertex (u, v) is called a *loop*.

A (di)graph G is called *trivial* if $|V(G)| = 1$. A graph (resp. digraph) G is called *complete* if there is an edge $\{u, v\}$ (i.e., a pair of edges (u, v) and (v, u)) for every two vertices $u, v \in V(G)$. A (di)graph G is called *bipartite* if $V(G)$ can be partitioned into two subsets, V_1 and V_2, so that every edge has one end vertex in V_1 and the other in V_2.

Undirected edges with the same pair of end vertices (or directed edges with the same tail and head) are called *multiple edges*. A graph (resp. digraph) is called a *multigraph* (a *multiple digraph*) if it is allowed to have multiple edges; otherwise it is called *simple*. We sometimes treat a multigraph G as a simple graph with integer-weighted edges, where the weight of each edge $e = \{u, v\}$ represents the number of multiple edges with the same end vertices u and v. In such an edge-weighted representation, the number m of edges means the number of pairs of adjacent vertices in G.

The *degree* of a vertex v in G is the number of edges incident with v. If G is a digraph, the *indegree* (resp. *outdegree*) denotes the number of edges incident to (resp. from) v. The minimum degree (resp. maximum degree) of the vertices in G is denoted by $\delta(G)$ (resp. $\Delta(G)$). An undirected graph (resp. digraph) G is called *Eulerian* if the degree of each vertex is even (resp. the indegree is equal to the outdegree at every vertex).

A graph $G' = (V', E')$ is called a *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$, which we denote by $G' \subseteq G$. G' is a *spanning subgraph* if $V' = V$. A subgraph $G' = (V', E')$ of $G = (V, E)$ is *induced* by V' if E' is given by $E' = \{e \in E \mid$ both end vertices of e belong to $V'\}$, and G' may be denoted by $G[V']$. Given an edge set F (not necessarily a subset of $E(G)$), we denote by $V[F]$ the set of end vertices of edges in F.

A sequence of vertices and edges in G, $P = (v_1, e_1, v_2, e_2, \ldots, e_{k-1}, v_k)$, is called a *path* between v_1 and v_k (or from v_1 to v_k if G is a digraph) if $v_1, v_2, \ldots, v_k \in V$, $e_1, e_2, \ldots, e_{k-1} \in E$ and $e_i = \{v_i, v_{i+1}\}$ (or $e_i = (v_i, v_{i+1})$ if G is a digraph), $i = 1, 2, \ldots, k - 1$. Such a path P is also denoted as a sequence of vertices $P = (v_1, v_2, \ldots, v_k)$ or a sequence of edges $P = (e_1, e_2, \ldots, e_{k-1})$ if

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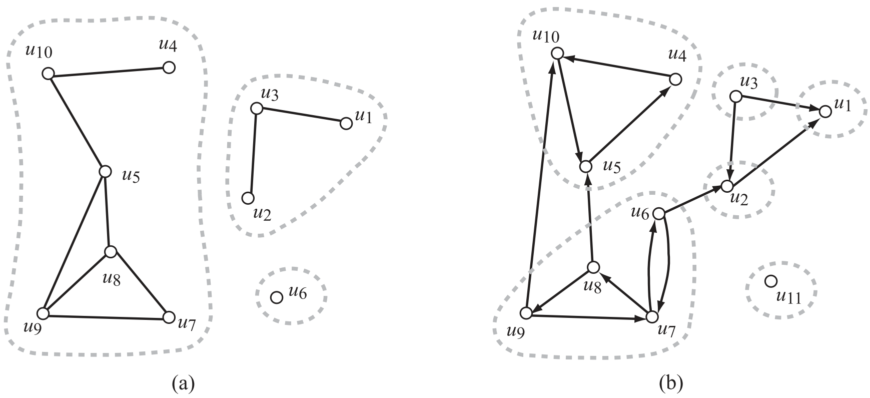

Figure 1.1. (a) A simple graph with three connected components; (b) a simple digraph with six strongly connected components, where each (strongly) connected component is enclosed by a gray dashed curve.

no confusion arises. For two vertices $u, v \in V$ in a graph (resp. digraph) G, a path between u and v (resp. a directed path from u to v) is called a (u, v)*-path*.

A graph (resp. digraph) G is called *connected* (resp. *strongly connected*) if G has a (u, v)-path for every pair of vertices u and v. A *connected component* (or a *component*) of a graph G is an inclusion-wise maximal vertex subset $X \subseteq V(G)$ such that every two vertices in X are connected by a path, where the induced subgraph $G[X]$ may also be called a (connected) component of G. A *strongly connected component* of a digraph G is an inclusion-wise maximal vertex subset $X \subseteq V(G)$ such that G has (u, v)- and (v, u)-paths for every two vertices u, $v \in X$, where the induced subgraph $G[X]$ may also be called a strongly connected component of G. Figure 1.1 illustrates examples of connected components of a graph and strongly connected components of a digraph.

An Eulerian connected graph has a sequence of edges by which we can visit all edges successively; we call such a sequence an *Eulerian trail*. Analogously an Eulerian strongly connected graph also admits an Eulerian trail, in which all directed edges are traversed along their directions. Figure 1.2(a) and (b) illustrate examples of Eulerian connected graph and strongly connected digraph, respectively.

1.1.1 Cut Functions of Weighted Graphs

When G is edge-weighted, the weight of an edge $e = \{u, v\}$ is denoted by $c_G(e)$ or $c_G(u, v)$, which are assumed to be nonnegative unless otherwise stated. Figure 1.3 shows a graph with integer edge weights, which can be viewed as a multigraph with multiplicity equal to the weight of each edge.

For two subsets $X, Y \subset V$ (not necessarily disjoint), $E(X, Y; G)$ denotes the set of edges e joining a vertex in X and a vertex in Y (i.e., $e = \{u, v\}$ satisfies $u \in X$ and $v \in Y$), and $d(X, Y; G)$ denotes $\sum_{e \in E(X,Y;G)} c_G(e)$. For a digraph $G = (V, E)$, we mean by $E(X, Y; G)$ the set of directed edges with a tail in X and

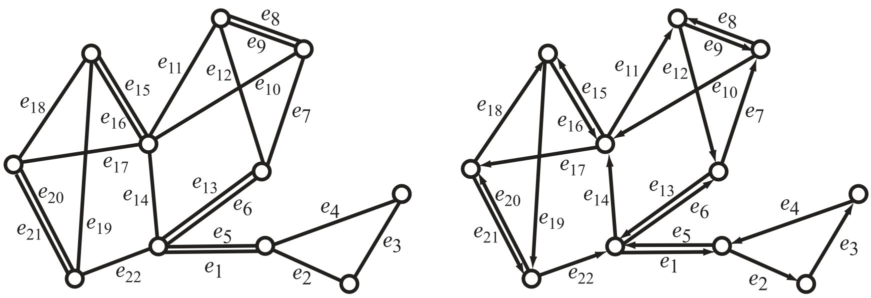

Figure 1.2. (a) An Eulerian connected graph; (b) an Eulerian strongly connected digraph, where the edges in the graph and digraph are indexed so that $(e_1, e_2, \ldots, e_{22})$ gives an Eulerian trail.

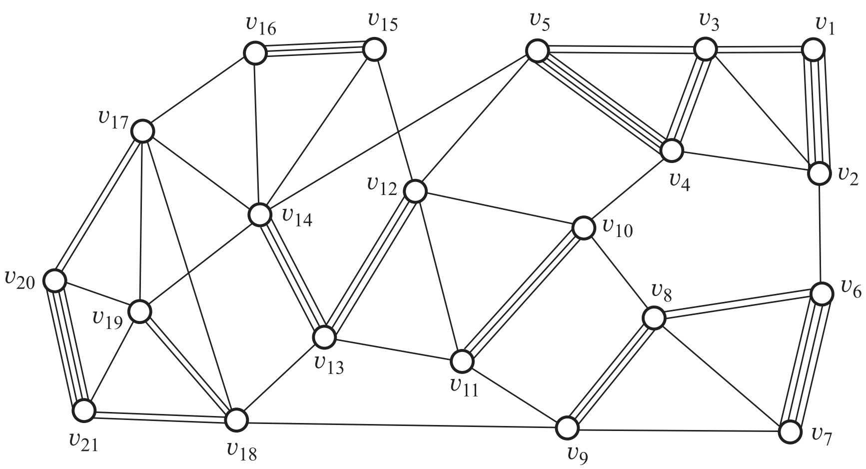

Figure 1.3. An integer-weighted graph G (the number of lines between two vertices represents the weight of the corresponding edge).

a head in Y. $E(X, Y; G)$ and $d(X, Y; G)$ may be written as $E(X; G)$ and $d(X; G)$, respectively, if $Y = V - X$, where $d(\emptyset; G) = d(V; G) = 0$ is assumed for convenience. In particular, the degree of v may be defined as $d(v; G)$. If G is clear from context, $E(X, Y; G)$ and $d(X, Y; G)$ may also be written as $E(X, Y)$ and $d(X, Y)$, respectively.

For a digraph $G = (V, E)$ and a subset $X \subseteq V$, we may write $d(X, V - X; G)$ and $d(V - X, X; G)$ as $d^+(X; G)$ and $d^-(X; G)$, respectively. The functions $d^+ : 2^V \to \Re_+$ with $d^+(X) = d^+(X; G)$, $X \in 2^V$, and $d^- : 2^V \to \Re_+$ with $d^-(X) = d^-(X; G)$, $X \in 2^V$, are called *cut functions* of a digraph G.

Lemma 1.1. *For two subsets X and Y of V in a digraph $G = (V, E)$, it holds that*

$$d^+(X; G) + d^+(Y; G) = d^+(X \cap Y; G) + d^+(X \cup Y; G) + d(X - Y, Y - X; G) + d(Y - X, X - Y; G) \quad (1.1)$$

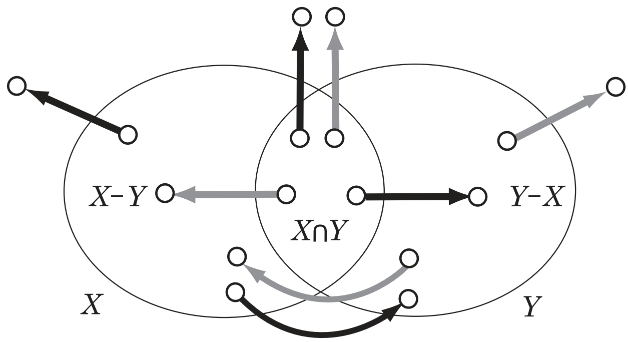

Figure 1.4. Illustration for $d^+(X;G)$ and $d^+(Y;G)$ of two subsets X and Y where black edges (resp. gray edges) represent $d^+(X;G)$ (resp. $d^+(Y;G)$).

and

$$d^-(X;G) + d^-(Y;G) = d^-(X \cap Y;G) + d^-(X \cup Y;G) \\ + d(X - Y, Y - X;G) + d(Y - X, X - Y;G). \quad (1.2)$$

□

Proof. For simplicity, we omit indicating G in $d(\cdot;G)$ and $d(\cdot,\cdot;G)$. We prove (1.1) ((1.2) can be treated symmetrically). The case where $X \subseteq Y$ or $Y \subseteq X$ is trivial since $\{X \cap Y, X \cup Y\} = \{X, Y\}$ and $\emptyset \in \{X - Y, Y - X\}$ hold. If $X \cap Y = \emptyset$, then we easily observe that $d^+(X) + d^+(Y) = d^+(X \cup Y) + d^+(X, Y) + d^+(Y, X)$ holds, implying (1.1). Consider the remaining case where X and Y intersect each other. In this case, we have

$$d^+(X \cap Y) = d(X \cap Y, X - Y) + d(X \cap Y, V - X)$$

and

$$d^+(X \cup Y) = d(Y, V - (X \cup Y)) + d(X - Y, V - (X \cup Y)).$$

See Fig. 1.4. On the other hand, we observe that the following equalities hold:

$$d^+(X) = d(X \cap Y, V - X) + d(X - Y, V - (X \cup Y)) + d(X - Y, Y - X),$$

$$d^+(Y) = d(X \cap Y, X - Y) + d(Y, V - (X \cup Y)) + d(Y - X, X - Y).$$

This proves the lemma. □

Let G be an undirected graph. The function $d : 2^V \to \Re_+$ with $d(X) = d(X;G)$ for all $X \in 2^V$ is called a *cut function* of G. Let G' be the digraph obtained from G by replacing each edge $\{u, v\} \in E$ with two oppositely oriented directed edges (u, v) and (v, u). Then $d(X;G) = d^+(X;G') = d^-(X;G')$ holds for all $X \subseteq V$. Hence, by Lemma 1.1, function $d : 2^V \to \Re_+$ satisfies the following inequality:

$$d(X;G) + d(Y;G) = d(X \cap Y;G) + d(X \cup Y;G) \\ + 2d(X - Y, Y - X;G). \quad (1.3)$$

By noting that $d(X;G) = d(V-X;G)$, we have $d(X;G)+d(Y;G) = d(V-X;G)+d(Y;G) = d((V-X)\cap Y;G)+d((V-X)\cup Y;G)+2d((V-X)-Y, Y-(V-X);G) = d(X-Y;G)+d(Y-X;G)+2d(X\cap Y, V-(X\cup Y);G)$. Thus,

$$d(X;G)+d(Y;G) = d(X-Y;G)+d(Y-X;G) + 2d(X\cap Y, V-(X\cup Y);G) \quad (1.4)$$

for all $X, Y \subseteq V$.

1.1.2 Vertex Neighbors

For a vertex $v \in V$ in an undirected graph G, let $\Gamma_G(v)$ denote the set of *neighbors* of v (i.e., vertices adjacent to v). For a subset $X \subseteq V$, let $\Gamma_G(X) = \cup_{v\in X}\Gamma_G(v) - X$, where $\Gamma_G(\emptyset) = \Gamma_G(V) = \emptyset$ is assumed for convenience. For a subset X of V in a digraph G, a vertex $v \in V-X$ is called an *out-neighbor* (resp. *in-neighbor*) of X if there is an edge $(z,v) \in E$ (resp. $(v,z) \in E$) for some $z \in X$. The set of all out-neighbors (resp. in-neighbors) of X is denoted by $\Gamma^+_G(X)$ (resp. $\Gamma^-_G(X)$). We call a subset $X \subseteq V$ *dominating* in G if $V-X-\Gamma^-_G(X) = \emptyset$ and *nondominating* if $V-X-\Gamma^-_G(X) \neq \emptyset$.

Lemma 1.2. *For two subsets X and Y of V in a digraph $G=(V,E)$, it holds that*

$$\begin{aligned}|\Gamma^+_G(X)|+|\Gamma^+_G(Y)| = {} & |\Gamma^+_G(X\cap Y)|+|\Gamma^+_G(X\cup Y)|\\ & +|(\Gamma^+_G(X-Y)-\Gamma^+_G(X\cap Y))\cap(Y-X)|\\ & +|(\Gamma^+_G(Y-X)-\Gamma^+_G(X\cap Y))\cap(X-Y)|\\ & +|(\Gamma^+_G(X-Y)\cap\Gamma^+_G(Y-X)-\Gamma^+_G(X\cap Y))\cap(V-(X\cup Y))|\end{aligned} \quad (1.5)$$

and

$$\begin{aligned}|\Gamma^-_G(X)|+|\Gamma^-_G(Y)| = {} & |\Gamma^-_G(X\cap Y)|+|\Gamma^-_G(X\cup Y)|\\ & +|(\Gamma^-_G(X-Y)-\Gamma^-_G(X\cap Y))\cap(Y-X)|\\ & +|(\Gamma^-_G(Y-X)-\Gamma^-_G(X\cap Y))\cap(X-Y)|\\ & +|(\Gamma^-_G(X-Y)\cap\Gamma^-_G(Y-X)\\ & -\Gamma^-_G(X\cap Y))\cap(V-(X\cup Y))|.\end{aligned} \quad (1.6)$$

□

Proof. We prove (1.5) ((1.6) can be treated symmetrically). From Fig. 1.5, we observe that

$$|\Gamma^+_G(X\cap Y)| = |\Gamma^+_G(X\cap Y)\cap(X-Y)|+|\Gamma^+_G(X\cap Y)\cap(V-X)|$$

and

$$\begin{aligned}|\Gamma^+_G(X\cup Y)| = {} & |\Gamma^+_G(Y)\cap(V-(X\cup Y))|\\ & +|(\Gamma^+_G(X-Y)-\Gamma^+_G(Y))\cap(V-(X\cup Y))|.\end{aligned}$$

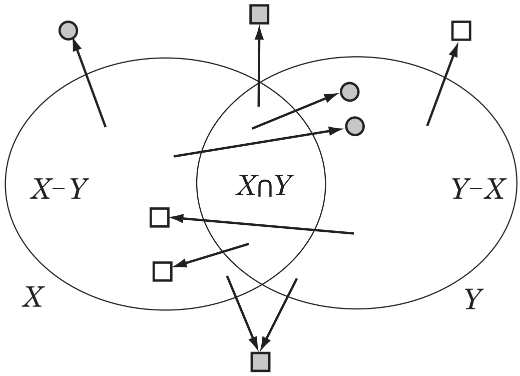

Figure 1.5. Illustration for $\Gamma_G^+(X)$ and $\Gamma_G^+(Y)$ of two intersecting subsets X and Y, where gray vertices represent $\Gamma_G^+(X)$ and square vertices represent $\Gamma_G^+(Y)$.

Furthermore, we observe that the following equalities also hold:

$$\begin{aligned}|\Gamma_G^+(X)| = {}& |\Gamma_G^+(X\cap Y)\cap(V-X)| + |(\Gamma_G^+(X-Y)-\Gamma_G^+(Y))\cap(V-(X\cup Y))| \\ &+ |(\Gamma_G^+(X-Y)\cap\Gamma_G^+(Y-X)-\Gamma_G^+(X\cap Y))\cap(V-(X\cup Y))| \\ &+ |(\Gamma_G^+(X-Y) - \Gamma_G^+(X\cap Y))\cap(Y-X)|,\end{aligned}$$

$$\begin{aligned}|\Gamma_G^+(Y)| = {}& |\Gamma_G^+(X\cap Y)\cap(X-Y)| + |\Gamma_G^+(Y)\cap(V-(X\cup Y))| \\ &+ |(\Gamma_G^+(Y-X)-\Gamma_G^+(X\cap Y))\cap(X-Y)|.\end{aligned}$$

From these equalities, we obtain (1.5). □

For an undirected graph $G = (V, E)$, let G' be the digraph obtained by replacing each edge $\{u, v\} \in E$ with two oppositely oriented directed edges, (u, v) and (v, u). Then $\Gamma_G(X) = \Gamma_{G'}^+(X) = \Gamma_{G'}^-(X)$ holds for all $X \subseteq V$. Hence, by Lemma 1.2, function $|\Gamma_G| : 2^V \to \mathbf{Z}_+$ satisfies the following inequality:

$$\begin{aligned}|\Gamma_G(X)| + |\Gamma_G(Y)| = {}& |\Gamma_G(X\cap Y)| + |\Gamma_G(X\cup Y)| \\ &+ |(\Gamma_G(X-Y)-\Gamma_G(X\cap Y))\cap(Y-X)| \\ &+ |(\Gamma_G(Y-X)-\Gamma_G(X\cap Y))\cap(X-Y)| \\ &+ |(\Gamma_G(X-Y)\cap\Gamma_G(Y-X) \\ &- \Gamma_G(X\cap Y))\cap(V-(X\cup Y))|.\end{aligned} \tag{1.7}$$

1.1.3 Graph Operations

For a subset $F \subseteq E$, $G - F$ denotes the graph obtained from G by removing the edges in F, and G/F denotes the graph obtained from G by contracting each edge $e \in F$ into a single vertex and deleting any resulting loops (and merging any resulting multiple edges into a single edge with the sum of their weights if G is an edge-weighted graph). See Fig. 1.6(b) and (c) for examples of $G - F$ and G/F. Note that G/F can be obtained in linear time by shrinking each connected component of (V, F) into a single vertex.

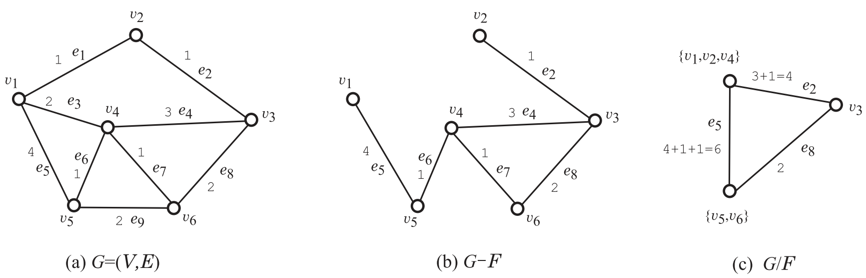

Figure 1.6. Illustration of graphs (b) $G - F$ and (c) G/F for (a) graph $G = (V, E)$ and for $F = \{e_1, e_3, e_9\}$ (the number beside each edge e indicates the weight of e).

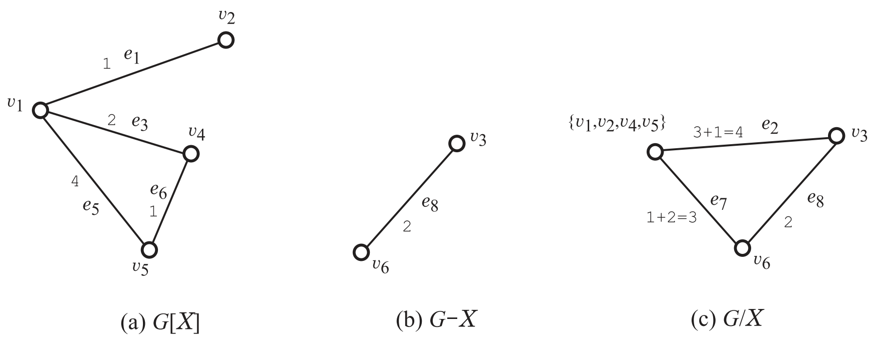

Figure 1.7. Illustration for graphs $G[X]$, $G - X$, and G/X of the graph $G = (V, E)$ in Fig. 1.6(a), where $X = \{v_1, v_2, v_4, v_5\}$.

For a set E' of new edges, we denote by $G + E'$ the graph obtained from G by adding the edges in E'.

Let X be a subset of V. We denote by $G[X]$ the subgraph induced from G by X, by $G - X$ the graph obtained from G by removing the vertices in X together with the edges incident with a vertex in X, and by G/X the graph obtained from G by contracting vertices in X into a single vertex and deleting any resulting loops (and merging any resulting multiple edges into a single edge with the sum of their weights if G is an edge-weighted graph). See Fig. 1.7(a), (b), and (c) as examples of $G[X]$, $G - X$, and G/X.

Given graph G, a *star augmentation* is a graph obtained by adding a new vertex s to G together with new weighted edges between s and some vertices in $V = V(G)$. A star augmentation H of G is defined by a vector $b \in \Re^V_+$ such that $b(v) = c_H(s, v)$ for each $v \in V$, where we let $b(v) = 0$ if edge $\{s, v\}$ is not introduced in the star augmentation. The star augmentation H defined by a vector $b \in \Re^V_+$ is denoted by $H = G + b$. See Fig. 1.8 for a star augmentation $H = G + b$ of graph $G = (V, E)$.

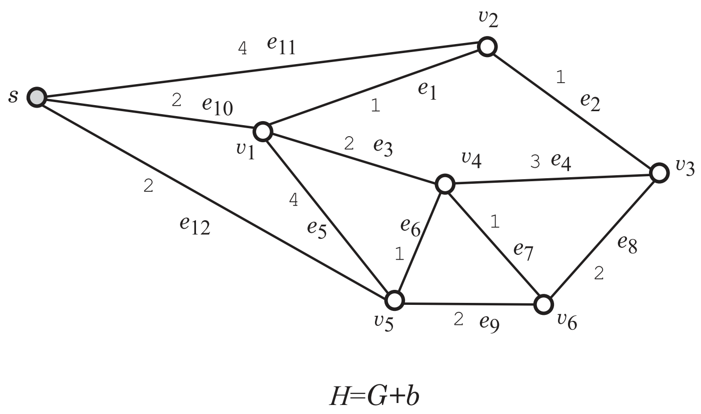

Figure 1.8. Illustration of the star augmentation $H = G + b$ of the graph $G = (V, E)$ defined in Fig. 1.6(a), where $b(v_1) = 2$, $b(v_2) = 4$, $b(v_5) = 2$, and $b(v_3) = b(v_4) = b(v_6) = 0$.

1.1.4 Edge-Connectivity

For an edge-weighted undirected graph (resp. digraph) G, a partition $\{X, V-X\}$ (resp. ordered partition $(X, V-X)$) of V, where X is a nonempty and proper subset X of V, is called a *cut* of G. A cut $\{X, V-X\}$ or $(X, V-X)$ is often denoted as X, for short, if no confusion arises. A subset $E' \subseteq E$ such that $E(X, V-X; G) \subseteq E'$ for some cut $\{X, V-X\}$ is called a *cut set* in G. An edge s is called a *cut edge* if $\{e\}$ is a cut set.

For a cut set E' such that $E' = E(X, V-X; G)$, we say that cut X is *generated* by E'. The *cut size* of a cut set E' (resp. cut X) is defined by $\sum_{e\in E'} c_G(e)$ (resp. $d(X; G)$). Let $S, T \subseteq V$. We say that a cut X *separates* S *and* T if $S \subseteq X \subseteq V-T$ or $T \subseteq X \subseteq V-S$, and that a cut X *separates* S *from* T if $S \subseteq X \subseteq V-T$. A cut separating S from T is called an (S, T)*-cut*. If S and T are singletons $\{s\}$ and $\{t\}$, respectively, we also refer to an (S, T)-cut as an (s, t)-cut. For two vertices, s and t, in an undirected graph/digraph G, an (s, t)-cut X (i.e., separating u and v) with the minimum cut size is called a *minimum* (s, t)-cut, and the cut size $d(X; G)$ is called *the local edge-connectivity* $\lambda(u, v; G)$ between u and v.

In an undirected graph/digraph G, we call a minimum (s, t)-cut X *minimal* (resp. *maximal*) if vertex set X is inclusion-wise minimal (resp. maximal) among all minimum (s, t)-cuts.

Lemma 1.3. *In an undirected graph/digraph G, a minimal* (*resp. maximal*) *minimum* (s, t)*-cut $X_{s,t}$ is unique; that is, $|X_{s,t}|$ is minimum* (*resp. maximum*) *among all minimum* (s, t)*-cuts.* □

Proof. We deal with the case of minimal minimum (s, t)-cuts in a digraph; the case of maximal minimum (s, t)-cuts or undirected graphs can be treated analogously.

Assume that there are two minimal minimum (s, t)-cuts, X and X'. Then X and X' cross each other. By (1.1), we have

$$2\lambda(s, t; G) = d^+(X; G) + d^+(X'; G) \geq d^+(X \cap X'; G) + d^+(X \cup X'; G).$$

By the minimality of X and X', $d^+(X \cap X'; G) > \lambda(s, t; G)$. Hence, we have $d^+(X \cup X'; G) < \lambda(s, t; G)$, contradicting that $d^+(Y) \geq \lambda(s, t; G)$ holds for all (s, t)-cuts Y. □

Given a subset $S \subseteq V$ and a vertex $v \in V - S$, we define the edge-connectivity $\lambda(S, v; G)$ by the minimum size of a cut X that separates S and v. Notice that $\lambda(S, v; G)$ is equal to $\lambda(s, v; G/S)$ of the graph G/S obtained by contracting S into a single vertex s.

The minimum cut size among all the cuts in G is called the *edge-connectivity* of G and is denoted by $\lambda(G)$. The edge connectivity of a graph consisting of a single vertex is set to be $+\infty$. Also by convention, $\lambda(v, v; G)$ is defined to be $+\infty$. These definitions tell that

$$\lambda(G) = \min_{u,v \in V} \lambda(u, v; G).$$

A cut X satisfying $d(X; G) = \lambda(G)$ is called a *minimum cut* in G. For a real $k \in \Re_+$, a graph G is *k-edge-connected* if $\lambda(G) \geq k$.

1.1.5 Vertex-Connectivity

For a connected undirected graph (resp. a strongly connected digraph) $G = (V, E)$, a subset $Z \subset V$ is called a *vertex cut* if $G - Z$ has at least two connected components (resp. at least two strongly connected components). The size of a vertex cut Z is defined by $|Z|$.

The minimum size of a vertex cut in G is called the (*global*) *vertex-connectivity* and is denoted by $\kappa(G)$. As a special case, we define the vertex-connectivity of a complete graph G by $\kappa(G) = n - 1$, where $n = |V(G)|$, since G has no vertex cut Z with $|Z| < n - 1$. A graph G is called *k-vertex-connected* if $n \geq k + 1$ and $\kappa(G) \geq k$ (i.e., there is no vertex cut S of size $k - 1$). By definition, $\kappa(G) \leq n - 1$ holds, where the equality holds only when G is a complete graph.

For two disjoint subsets $A, B \subseteq V$, we say that a subset $C \subset V - (A \cup B)$ *separates* A *from* B in G if $G - Z$ has no (u, v)-path for any pair $u \in A$ and $v \in B$. A vertex cut Z separating A from B is called an (A, B)*-vertex cut*. A vertex v is called a *cut vertex* if $\{v\}$ is a vertex cut.

Given two vertices, u and v, a set of (u, v)-paths is called *internally vertex-disjoint* if no two of them share any vertex other than u and v. *The local vertex-connectivity* between vertices u and v is defined to be the maximum number of internally vertex-disjoint (u, v)-paths and is denoted by $\kappa(u, v; G)$. As will be seen in Section 1.4, if $\{u, v\} \notin E$ in a graph (resp. $(u, v) \notin E$ in a digraph) G, then $\kappa(u, v; G)$ is equal to the minimum size $|Z|$ of a (u, v)-vertex cut Z. Observe