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# Difference Equations by Differential Equation Methods

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To

Alison,
Chris,
Rachel
and
Katy,

who didn't believe me
when I last said "Never again".

You were right.

# Contents

# Preface

Difference equations are prevalent in mathematics, occurring in areas as disparate as number theory, control theory and integrable systems theory. They arise as mathematical models of discrete processes, as interesting dynamical systems, and as finite difference approximations to differential equations. Finite difference methods exploit the fact that differential calculus is a limit of the calculus of finite differences. It is natural to take this observation a step further and ask whether differential and difference equations share any common features. In particular, can they be solved by the same (or similar) methods?

Just over twenty years ago, a leading numerical analyst summarized the state of the art as follows: problems involving difference equations are an order of magnitude harder than their counterparts for differential equations. There were two major exceptions to this general rule. Linear ordinary difference equations behave similarly to their continuous counterparts. (Indeed, most of the best-known texts on difference equations deal mainly with linear and linearizable problems.) Discrete integrable systems are nonlinear, but have some underlying linear structures; they have much in common with continuous integrable systems, together with some interesting extra features.

Research since that time has transformed our understanding of more general difference equations and their solutions. The basic geometric structures that underpin difference equations are now known. From these, it has been possible to develop a plethora of systematic techniques for finding solutions, first integrals or conservation laws of a given difference equation. These look a little different to the corresponding methods for differential equations, mainly because the solutions of difference equations are not continuous. However, many techniques are widely applicable and most do not require the equation to have any special properties such as linearizability or integrability.

This book is intended to be an accessible introduction to techniques for general difference equations. The basic geometrical structures that lie behind these techniques are also described. My aim throughout has been to describe ideas that are applicable to all sufficiently well-behaved difference equations, without assuming prior knowledge of any special properties that an equation may possess. There is an exception: the last chapter includes material on Noether's Theorem and several related results, which apply only to difference equations that stem from a variational problem. As variational problems are commonplace, I think that many readers will find this material interesting and useful. I have not included material on the qualitative theory of difference equations. In particular, there is no discussion of stability or oscillation theory. I recommend Elaydi (2005) as an excellent introduction to these topics.

The book may be used as the basis for an advanced undergraduate or postgraduate course on difference equations; the main prerequisite is a working knowledge of solution methods for differential equations. It is also designed for self-study. Most of the material is presented informally, with few theorems and many worked examples; a triangle (▲) separates each example from the subsequent text. To help readers grasp the basic concepts and tools early on, each major idea is introduced in its simplest context, leaving generalizations until later. Every chapter concludes with a range of exercises of varying difficulty. Most are designed to enable readers to become proficient in using techniques; a few develop extensions of the core material. Where an exercise is particularly challenging, this is indicated by an asterisk (*).

Each chapter includes suggestions for further reading. These are works that will introduce readers to some topics in fields that are beyond the scope of this book (such as discrete integrable systems and geometric integration). I have not surveyed the literature on any specialized topic, however, in order to keep the book focused on methods that apply to most types of difference equations.

The first three chapters introduce the main ideas and methods in their simplest setting: ordinary difference equations (OΔEs). Together, these chapters include more than enough material for a single-semester course for advanced undergraduates. Chapter 1 surveys a range of methods for solving linear OΔEs. Its purpose is to give the beginner a rapid introduction to standard techniques, though some novel methods are also included. Chapter 2 introduces an indirect way to solve or simplify a given nonlinear OΔE, by first solving an associated linear problem. This approach, called symmetry analysis, lies at the heart of most exact methods for solving nonlinear ordinary differential equations. A dual approach is to construct first integrals; this too has a counterpart for OΔEs. There are various extensions of basic symmetry methods for OΔEs; some of the most useful ones are described in Chapter 3.

The pace is quickened somewhat in the second part of the book, which brings the reader to the leading edge of research. The focus is mainly on partial difference equations (PΔEs), starting in Chapter 4 with a detailed description of their underlying geometry. A key idea is that transforming a given PΔE (or writing it in different coordinates) may enable one to understand it better. In particular, one can identify circumstances under which an initial-value problem has a unique solution; this is a prerequisite for symmetry methods to work. Chapter 5 describes various methods for obtaining exact solutions of a given PΔE. These include some well-established techniques for linear PΔEs, together with symmetry analysis and other recent methods for nonlinear PΔEs.

Often, one is interested in properties that are shared by all solutions of a given partial differential equation, particularly conservation laws. Famously, Noether's Theorem uses symmetries of an underlying variational problem to generate conservation laws. There is also a way to construct conservation laws directly, whether or not a variational formulation is known. Chapter 6 describes PΔE analogues of Noether's Theorem and the direct method. Noether's paper on variational symmetries includes a second theorem, which identifies relationships between the Euler–Lagrange equations for gauge theories. Again, it turns out that there is a difference version of this result. The book concludes with a brief outline of the reason for the close analogy between conservation laws of differential and difference equations, with suggestions for further reading.

Difference equations have a well-deserved reputation for being tricky. Yet many of them are susceptible to differential equation techniques, provided that these are used with care. My aim has been to give the reader practical tools, coupled with sufficient geometrical understanding to use them correctly.

*Peter E. Hydon*

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