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1

Elementary methods for linear OΔEs

As Shakespeare says, if you're going to do a thing
you might as well pop right at it and get it over.

(P. G. Wodehouse, Very Good, Jeeves!)

This chapter is a rapid tour through the most useful methods for solving scalar linear ordinary difference equations (OΔEs). *En route*, we will see some differences between OΔEs and ordinary differential equations (ODEs). Yet what is most striking is how closely methods for linear OΔEs correspond to their counterparts for linear ODEs. Much of this similarity is due to linearity.

1.1 Basic definitions and notation

An OΔE or system of OΔEs has a single integer-valued independent variable, n, that can take any value within a domain $D \subset \mathbb{Z}$. A *scalar* OΔE (also called a *recurrence relation*) has just one dependent variable, u, which we shall assume is real-valued. The OΔE is *linear* if it can be written in the form

$$a_p(n)\,u(n+p) + a_{p-1}(n)\,u(n+p-1) + \cdots + a_0(n)\,u(n) = b(n), \qquad (1.1)$$

where p is a positive integer and each a_i is a given real-valued function. The *order* of the OΔE at n is the difference between the highest and lowest arguments of u in (1.1). A point $n \in D$ is a *regular point* of the OΔE if $a_p(n)$ and $a_0(n)$ are both nonzero; otherwise it is *singular*. The OΔE (1.1) is of order p only at regular points; it is of lower order at singular points.

Example 1.1 The following OΔE has two singular points:

$$(n-1)^2 u(n+2) + 2\,u(n+1) + (1-n^2)\,u(n) = 0, \qquad n \in \mathbb{Z}.$$

At $n = -1$, the OΔE reduces to the first-order equation $4\,u(1) + 2\,u(0) = 0$. At $n = 1$, it amounts to $2\,u(2) = 0$, which is not even a difference equation. ▲

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Example 1.2 As an extreme example, consider the OΔE

$$(1 + (-1)^n)\, u(n + 2) - 2u(n + 1) + (1 - (-1)^n)\, u(n) = 0, \qquad n \in \mathbb{Z}. \quad (1.2)$$

Although this looks like a second-order OΔE, *every* point is singular! Furthermore, (1.2) yields exactly the same equation when $n = 2m$ as it does when $n = 2m + 1$, namely

$$u(2m+2) = u(2m+1).$$ ▲

Given an OΔE (1.1), we call D a *regular domain* if it is a set of consecutive regular points. Suppose that $\mathbb{Z}$ is a regular domain, so that $a_p(n)$ and $a_0(n)$ are nonzero for every integer n. Suppose also that, for some n_0, the p consecutive values $u(n_0), \ldots, u(n_0+p-1)$ are known. Then one can calculate $u(n_0+p)$ by setting $n = n_0$ in (1.1). Repeating this process, using $n = n_0+1, n_0+2, \ldots$ in turn, produces $u(n_0+i)$ for all $i > p$. The remaining values of u can also be obtained by setting $n = n_0-1, n_0-2$, and so on. Thus p (arbitrary) consecutive conditions determine a unique solution of the OΔE. This result holds for any regular domain; consequently, the general solution of every p^{th}-order linear OΔE on a regular domain depends upon p arbitrary constants.

When the domain is not regular, various oddities may occur. For instance, the general solution of (1.2) depends on infinitely many arbitrary constants, because for each $m \in \mathbb{Z}$, one of the pair $\{u(2m+1), u(2m+2)\}$ must be given in order to determine the other. To avoid having to deal separately with singularities, we will ensure that D is regular from here on, setting $a_p(n) = 1$ without loss of generality; under these conditions, we describe the OΔE (1.1) as being in *standard form*.

For a p^{th}-order linear ODE,

$$y^{(p)}(x) + a_{p-1}(x)\, y^{(p-1)}(x) + \cdots + a_0(x)\, y(x) = b(x),$$

the usual convention suppresses the argument x where this can be assumed, leading to a slightly shorter expression:

$$y^{(p)} + a_{p-1}(x)\, y^{(p-1)} + \cdots + a_0(x)\, y = b(x).$$

Similarly, it is helpful to write the OΔE (1.1), in standard form, as

$$u_p + a_{p-1}(n)\, u_{p-1} + \cdots + a_0(n)\, u = b(n), \qquad (1.3)$$

where u and u_i are shorthand for $u(n)$ and $u(n+i)$, respectively. Suppressing the independent variable(s) saves considerable space, particularly for partial difference equations[1], so we will do this for all *unknown* functions of n. If a

[1] For instance, if u depends on $\mathbf{n} = (n^1, n^2, n^3, n^4)$ then $u(n^1 + 1, n^2 + 3, n^3 + 2, n^4 + 1)$ is written as $u_{1,3,2,1}$.

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function f is given, or is assumed to be known, we use $f(n)$, $f(n+1)$, and so on. In particular, a solution of the OΔE (1.3) will be an expression of the form

$$u = f(n),$$

where, for each $n \in D$, $u(n) = f(n)$ satisfies (1.1) with $a_p(n) = 1$.

If $b(n) = 0$ for each $n \in D$ then (1.3) is *homogeneous*; otherwise, (1.3) is *inhomogeneous* and its associated homogeneous equation is

$$u_p + a_{p-1}(n)\, u_{p-1} + \cdots + a_0(n)\, u = 0. \tag{1.4}$$

These definitions correspond to those that are used for linear ODEs. Just as for ODEs, there is a principle of linear superposition: if $u = f_1(n)$ and $u = f_2(n)$ are any two solutions of (1.4) then

$$u = c_1 f_1(n) + c_2 f_2(n) \tag{1.5}$$

is a solution for all constants c_1 and c_2. Henceforth, the notation c_i and $\tilde{c}_i$ will be reserved for arbitrary constants. (Sometimes it is convenient to replace one set of arbitrary constants, $\{c_i\}$, by another set, $\{\tilde{c}_i\}$.)

Suppose that p solutions of (1.4), $u = f_i(n)$, $i = 1, \ldots, p$, are linearly independent[2] on D, which means that

$$\sum_{i=1}^{p} c_i f_i(n) = 0, \text{ for all } n \in D \text{ if and only if } c_1 = c_2 = \cdots = c_p = 0.$$

Then the principle of linear superposition implies that

$$u = \sum_{i=1}^{p} c_i f_i(n) \tag{1.6}$$

is a solution of (1.4) for each choice of values for the p arbitrary constants c_i; consequently (1.6) is the general solution of (1.4). If, in addition, $u = g(n)$ is any particular solution of a given inhomogeneous OΔE (1.3), the general solution of that OΔE is

$$u = g(n) + \sum_{i=1}^{p} c_i f_i(n). \tag{1.7}$$

These results are proved in the same way as their counterparts for ODEs.

[2] A simple test for linear independence is given in Exercise 1.1.

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1.2 The simplest OΔEs: solution by summation

The starting-point for the solution of ODEs is the Fundamental Theorem of Calculus, namely

$$\int_{x=x_0}^{x_1} y'(x)\,\mathrm{d}x = y(x_1) - y(x_0).$$

A similar result holds for difference equations. Define the action of the *forward difference operator* Δ_n on any function $f(n)$ by

$$\Delta_n f(n) \equiv f(n+1) - f(n), \qquad \text{for all } n \in \mathbb{Z}. \tag{1.8}$$

In particular,

$$\Delta_n u \equiv u_1 - u \quad \text{and} \quad \Delta_n u_i \equiv u_{i+1} - u_i, \quad i \in \mathbb{Z}. \tag{1.9}$$

By summing consecutive differences, we obtain

$$\sum_{k=n_0}^{n_1-1} \Delta_k f(k) = \sum_{k=n_0}^{n_1-1} (f(k+1) - f(k)) = f(n_1) - f(n_0). \tag{1.10}$$

This very useful result is known as the *Fundamental Theorem of Difference Calculus*. We can use it immediately to solve OΔEs of the form

$$u_1 - u = b(n), \qquad n \geq n_0. \tag{1.11}$$

Replacing n by k in (1.11) and (f, n_1) by (u, n) in (1.10) yields

$$u = u(n_0) + \sum_{k=n_0}^{n-1} b(k); \tag{1.12}$$

we adopt the convention that a sum is zero if its lower limit exceeds its upper limit, which occurs here only when $n = n_0$. The OΔE (1.11) is recovered by applying the forward difference operator Δ_n to (1.12):

$$\Delta_n u = \left(u(n_0) + \sum_{k=n_0}^{n} b(k) \right) - \left(u(n_0) + \sum_{k=n_0}^{n-1} b(k) \right) = b(n).$$

If the initial condition $u(n_0)$ is known, (1.12) is the unique solution of (1.11) that satisfies this initial condition; otherwise $u(n_0)$ in (1.12) can be replaced by an arbitrary constant c_1, which yields the general solution of (1.11). For instance, the general solution of the simplest OΔE, $\Delta_n u = 0$, on any regular domain is $u = c_1$.

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1.2.1 Summation methods

The solution (1.12) involves an unevaluated sum over multiple points k. If the sum is written as a function that is evaluated at n only, the solution is said to be in *closed form*. To obtain a closed-form solution of the OΔE (1.11), we need to find an *antidifference* (or indefinite sum) of the function $b(k)$, which is any function $B(k)$ that satisfies

$$\Delta_k B(k) = b(k) \quad \text{for every } k \in D. \tag{1.13}$$

If $B(k)$ is an antidifference of $b(k)$, so is $B(k) + c$ for any real constant c. Moreover, $u = B(n)$ is a particular solution of (1.11), so the general solution of this OΔE is

$$u = B(n) + \tilde{c}_1.$$

Indeed, if we substitute (1.13) into the sum in (1.12) and use the Fundamental Theorem of Difference Calculus, we obtain

$$u = B(n) - B(n_0) + u(n_0).$$

Summation is the difference analogue of integration, so antidifferences are as useful for solving OΔEs as antiderivatives (or indefinite integrals) are for solving ODEs. Table 1.1 lists some elementary functions and their antidifferences; these are sufficient to deal with the most commonly occurring sums. The functions $k^{(r)}$ in Table 1.1 are defined as follows:

$$k^{(r)} = \begin{cases} \dfrac{k!}{(k-r)!}, & k \geq r, \\ 0, & k < r. \end{cases} \tag{1.14}$$

The formula for the antidifference of $k^{(r)}$ looks very like the formula for the indefinite integral of x^r; it is easy to evaluate sums of these functions, as we are on familiar ground. In particular, we can sum all polynomials in (non-negative) k, because these can be decomposed into a sum of the functions

$$k^{(r)} = k(k-1)\cdots(k-r+1), \qquad 0 \leq r \leq k.$$

Example 1.3 Evaluate $\sum_{k=1}^{n-1} k^2$ and $\sum_{k=1}^{n-1} k^3$ in closed form.

Solution: Write $k^2 = k(k-1) + k = k^{(2)} + k^{(1)}$; then row **3** of Table 1.1 yields

$$k^2 = \Delta_k \left(\frac{1}{3} k^{(3)} + \frac{1}{2} k^{(2)} \right).$$

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Table 1.1 *Antidifferences of some elementary functions:* $\Delta_k(B(k)) = b(k)$.

	Function $b(k)$	Antidifference $B(k)$
1	1	k
2	k	$k(k-1)/2$
3	$k^{(r)},\ (r \neq -1, k \geq 0)$	$k^{(r+1)}/(r+1)$
4	$a^k,\ (a \neq 1)$	$a^k/(a-1)$
5	$\cos(ak+b),\ (a \not\equiv 0, \bmod 2\pi)$	$\sin(ak+b-a/2)/(2\sin(a/2))$
6	$\sin(ak+b),\ (a \not\equiv 0, \bmod 2\pi)$	$-\cos(ak+b-a/2)/(2\sin(a/2))$

Now sum to get

$$\sum_{k=1}^{n-1} k^2 = \left[\frac{1}{3}\,k^{(3)} + \frac{1}{2}\,k^{(2)}\right]_{k=1}^{n} = \frac{1}{6}\,n(n-1)(2n-1).$$

Similarly, the decomposition $k^3 = k^{(3)} + 3k^{(2)} + k^{(1)}$ yields

$$\sum_{k=1}^{n-1} k^3 = \left[\frac{1}{4}\,k^{(4)} + k^{(3)} + \frac{1}{2}\,k^{(2)}\right]_{k=1}^{n} = \frac{1}{4}\,n^2(n-1)^2. \qquad \blacktriangle$$

The functions $k^{(r)}$ are also useful for summing some rational polynomials because, for $r < 0$,

$$k^{(r)} = \frac{1}{(k+1)(k+2)\cdots(k-r)}\,. \tag{1.15}$$

Example 1.4 Calculate $\displaystyle\sum_{k=1}^{n-1} \frac{k}{(k+1)(k+2)(k+3)}$.

Solution: First decompose the summand as follows:

$$\frac{k}{(k+1)(k+2)(k+3)} = \frac{1}{(k+1)(k+2)} - \frac{3}{(k+1)(k+2)(k+3)} = k^{(-2)} - 3k^{(-3)}.$$

Then use row **3** of Table 1.1 to obtain

$$\sum_{k=1}^{n-1} \frac{k}{(k+1)(k+2)(k+3)} = \left[-k^{(-1)} + \frac{3}{2}\,k^{(-2)}\right]_{k=1}^{n} = \frac{n(n-1)}{4(n+1)(n+2)}\,. \qquad \blacktriangle$$

The method used in this example works for any rational polynomial of the form $k^{(r)}P_m(k)$, where $r \leq -2$ and $P_m(k)$ is a polynomial of degree $m \leq -r-2$. The reason why $P_m(k)$ must be at least two orders lower than $1/k^{(r)}$ is that terms

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proportional to $k^{(-1)}$ cannot be evaluated in closed form. The sums

$$H_n = \sum_{k=0}^{n-1} k^{(-1)} = \sum_{k=1}^{n} \frac{1}{k}$$

are called *harmonic numbers*. For large n, the harmonic numbers are approximated to $O(n^{-2})$ by $H_n \approx \ln(n) + \gamma + 1/(2n)$; here $\gamma \approx 0.5772$ is the Euler constant. A better estimate, whose error is $O(n^{-4})$, is

$$H_n \approx \frac{\ln(n+1) + \ln(n)}{2} + \gamma + \frac{1}{6n(n+1)}.$$

A sum whose summand is a periodic function of k may be expressed in closed form as follows. Calculate the contribution per period and multiply this by the number of complete periods in the range; then add any remaining terms. If the period is small, this is often the simplest way of evaluating such sums. Alternatively, one can write the periodic function as a sum of sines and cosines and then use rows **5** and **6** of Table 1.1.

Example 1.5 Calculate $\sum_{k=n}^{0} (-1)^{k(k-1)/2}$, where $n < 0$.

Solution: Note that $(-1)^{k(k-1)/2} = \sqrt{2}\sin((2k+1)\pi/4)$; so row **6** gives

$$\sum_{k=n}^{0} (-1)^{k(k-1)/2} = \left[-\cos(k\pi/2)\right]_{k=n}^{1} = \cos(n\pi/2). \qquad \blacktriangle$$

These methods of summation can be combined to deal with summands that are products of powers, polynomials and periodic functions. It would be convenient if there were an analogue of the Leibniz product rule (which leads directly to the formula for integration by parts). Unlike the differential operator $\mathrm{d}/\mathrm{d}x$, however, the forward difference operator does not satisfy the Leibniz product rule, because

$$\begin{aligned} \Delta_n\{f(n)\,g(n)\} &= f(n+1)\,g(n+1) - f(n)\,g(n) \\ &\neq \Delta_n\{f(n)\}g(n) + f(n)\,\Delta_n\{g(n)\}. \end{aligned}$$

Instead, the following modified Leibniz rule holds:

$$\Delta_n\{f(n)\,g(n)\} = \Delta_n\{f(n)\}\,g(n+1) + f(n)\,\Delta_n g(n). \tag{1.16}$$

This leads to the extraordinarily useful *summation by parts* formula,

$$\sum_{k=n_0}^{n_1-1} f(k)\,\Delta_k g(k) = \left[f(k)\,g(k)\right]_{k=n_0}^{n_1} - \sum_{k=n_0}^{n_1-1} \{\Delta_k f(k)\}\,g(k+1). \tag{1.17}$$

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Example 1.6 Calculate $\sum_{k=1}^{n-1} ka^k$, where $a \neq 1$.

Solution: Substitute $f(k) = k,\ \ g(k) = a^k/(a-1)$ into (1.17) to obtain

$$\sum_{k=1}^{n-1} ka^k = \left[\frac{ka^k}{a-1}\right]_{k=1}^{n} - \sum_{k=1}^{n-1} \frac{a^{k+1}}{a-1} = \left[\frac{ka^k}{a-1} - \frac{a^{k+1}}{(a-1)^2}\right]_{k=1}^{n} = \frac{na^n}{a-1} - \frac{a^{n+1}-a}{(a-1)^2}.$$

▲

These elementary summation techniques are sufficient to deal with most simple problems; however, there are many functions whose sum cannot be expressed in closed form. Nevertheless, we will regard an OΔE as being solved when its general solution is given, even if this is in terms of one or more unevaluated sums.

1.2.2 The summation operator

So far, we have restricted attention to OΔEs for which $n \geq n_0$. If solutions are sought for all $n \in \mathbb{Z}$, or for $n < n_0$, it is also necessary to solve

$$u_1 - u = b(n), \qquad n < n_0. \tag{1.18}$$

To do this, use (1.10) with n_0 and n_1 replaced by n and n_0 respectively, which yields

$$u = u(n_0) - \sum_{k=n}^{n_0-1} b(k). \tag{1.19}$$

It is helpful to combine (1.12) and (1.19) by defining the *summation operator* $\boldsymbol{\sigma}_k$ as follows:

$$\boldsymbol{\sigma}_k\{f(k); n_0, n_1\} = \begin{cases} \displaystyle\sum_{k=n_0}^{n_1-1} f(k), & n_1 > n_0; \\ 0, & n_1 = n_0; \\ \displaystyle -\sum_{k=n_1}^{n_0-1} f(k), & n_1 < n_0. \end{cases} \tag{1.20}$$

This operator satisfies the identity

$$\boldsymbol{\sigma}_k\{f(k); n_0, n+1\} = \boldsymbol{\sigma}_k\{f(k); n_0, n\} + f(n). \tag{1.21}$$

So, given $u(n_0)$, the solution of

$$\Delta_n u = b(n), \qquad n \in \mathbb{Z}, \tag{1.22}$$

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that satisfies the initial condition is

$$u = u(n_0) + \boldsymbol{\sigma}_k\{b(k); n_0, n\}. \tag{1.23}$$

If no initial condition is prescribed, the general solution of (1.22) is

$$u = c_1 + \boldsymbol{\sigma}_k\{b(k); n_0, n\}, \tag{1.24}$$

where n_0 is any convenient integer in the domain. In accordance with the principle of linear superposition, $u = c_1$ is the general solution of the associated homogeneous equation, $\Delta_n u = 0$, and $u = \boldsymbol{\sigma}_k\{b(k); n_0, n\}$ is a particular solution of (1.22).

1.3 First-order linear OΔEs

The standard form of any first-order linear homogeneous OΔE is

$$u_1 + a(n)\,u = 0, \qquad a(n) \neq 0, \quad n \in D. \tag{1.25}$$

As in the last section, we begin by solving the OΔE on $D = \{n \in \mathbb{Z} : n \geq n_0\}$ before generalizing the result to arbitrary regular domains. To find u for $n > n_0$, replace n by $n-1$ in (1.25) and rearrange the OΔE as follows:

$$u = -a(n-1)\,u_{-1}. \tag{1.26}$$

Then replace n by $n-1, n-2, \ldots, n_0+1$ successively in (1.26) to obtain

$$\begin{aligned} u &= (-1)^2 a(n-1)\,a(n-2)\,u_{-2} \\ &= (-1)^3 a(n-1)\,a(n-2)\,a(n-3)\,u_{-3} \\ &\;\;\vdots \\ &= (-1)^{n-n_0} a(n-1)\,a(n-2)\,a(n-3)\cdots a(n_0)\,u(n_0). \end{aligned}$$

Therefore the solution of (1.25) for $n > n_0$ is

$$u = u(n_0)\prod_{k=n_0}^{n-1}(-\,a(k)) = u(n_0)(-1)^{n-n_0}\prod_{k=n_0}^{n-1}a(k). \tag{1.27}$$

If no initial condition is prescribed, $u(n_0)$ is replaced by an arbitrary constant.

Example 1.7 Solve the initial-value problem

$$u_1 - 3u = 0, \qquad n \geq 0, \qquad \text{subject to} \quad u(0) = 2.$$

Solution: Substitute $a(n) = -3$, $n_0 = 0$ and $u(n_0) = 2$ into (1.27) to obtain

$$u = 2\prod_{k=0}^{n-1} 3 = 2 \times 3^n.$$

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Example 1.8 Let α be a positive constant. Find the general solution of

$$u_1 - (n+\alpha)\,u = 0, \qquad n \geq 0.$$

Solution: Here $a(n) = -(n+\alpha)$ and $n_0 = 0$, so (1.27) yields

$$u = c_1 \prod_{k=0}^{n-1}(k+\alpha) = \frac{c_1\Gamma(n+\alpha)}{\Gamma(\alpha)}\,.$$

Here $\Gamma(z)$ is the *gamma function* (the analytic continuation of the factorial function to the complex plane with the non-positive integers $0, -1, -2, \ldots$ deleted). The gamma function satisfies

$$\Gamma(z+1) = z\,\Gamma(z); \tag{1.28}$$

for $\mathrm{Re}\{z\} > 0$, it is defined by the integral

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}t. \tag{1.29}$$

A simple integration by parts shows that the definition (1.29) is consistent with (1.28). The gamma function may also be evaluated in the region $\mathrm{Re}\{z\} \leq 0$, except at the deleted points. This is done by using (1.28) repeatedly to find $\Gamma(z)$ in terms of $\Gamma(z+N)$, where $N \in \mathbb{N}$ is large enough to make $\mathrm{Re}\{z+N\}$ positive. Two immediate consequences of (1.28) and (1.29) are the useful identities

$$\Gamma(n+1) = n! \quad \text{for all } n \in \mathbb{N}, \qquad\qquad \Gamma(1/2) = \sqrt{\pi}\,. \qquad \blacktriangle$$

It is worth observing that (1.27) is closely related to the formula (1.12) for inverting the forward difference operator. For simplicity, suppose that $a(n) < 0$ for every $n \geq n_0$ and that $u(n_0) > 0$; this guarantees that u is positive throughout the domain. Then the logarithm of (1.26), with n replaced by $n+1$, amounts to the O∆E

$$\Delta_n(\ln u) = \ln\left(-\,a(n)\right), \qquad n \geq n_0. \tag{1.30}$$

From (1.12), the general solution of (1.30) is

$$\ln u = \ln\left(u(n_0)\right) + \sum_{k=n_0}^{n-1} \ln\left(-\,a(k)\right) = \ln\left(u(n_0)\prod_{l=n_0}^{n-1}\left(-\,a(l)\right)\right),$$

which is the logarithm of (1.27). In other words, we could have solved (1.25) by transforming it to an O∆E of a type that we already know how to solve. The same approach lies at the heart of many methods for O∆Es and ODEs alike: transform them to something simpler. Later in this chapter, we will consider

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