1
Action principle in classical mechanics

1.1 Euler–Lagrange equations

In classical mechanics we are usually interested in solving the equations of motion of a point particle under the action of some prescribed force. A method for doing this in a systematic way is given by Newton's equation $\mathbf{F} = m\mathbf{a}$, where $\mathbf{F}$ is the force applied, m is the mass, and $\mathbf{a}$ is the acceleration.[1] Since the force is often a function of time t and the position $\mathbf{x}$ of the particle (it may also depend on velocity in some cases), and the acceleration $\mathbf{a}$ is given in terms of the vector $\mathbf{x}$ specifying the position of the particle by $\mathbf{a} = \ddot{\mathbf{x}}$, Newton's equations of motion result in a set of ordinary differential equations for the path $\mathbf{x}(t)$ followed by the particle. The motion is completely specified if we give the position $\mathbf{x}(t_0) = \mathbf{x}_0$ and the velocity $\dot{\mathbf{x}}(t_0) = \mathbf{v}_0$ at some initial time t_0. (Other conditions on the path are also possible. For example, we might specify the location of the particle at two different times.)

It is usually assumed that the path of the particle can be described by regarding $\mathbf{x}$ as a vector in the Euclidean space $\mathbb{R}^D$, and typically we are interested in the case $D = 3$. The components of $\mathbf{x}$ may be chosen to be the Cartesian coordinates of the curve $\mathbf{x}(t)$ that describes the motion of the particle parameterized by the time t. However, the choice of Cartesian coordinates is totally arbitrary, and any coordinates may be used.[2] This basic principle of relativity is used almost without comment all the time. Two independent observers looking at the same particle under the action

[1] Here and throughout boldfaced text will be used to denote a vector.
[2] The physics of the situation must be independent of this arbitrary choice of coordinates.

of a given force and who adopt different choices of coordinates should agree on the behaviour of the particle in any situation.[3]

We are often interested in a conservative force which can be derived from a potential using $\mathbf{F} = -\nabla V$. For such forces the work done on a particle moving in the field of force between two points in space, $\mathbf{x}_1$ and $\mathbf{x}_2$, is

$$\text{Work}(1 \to 2) = \int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{F} \cdot d\mathbf{x} = -(V_2 - V_1).$$

The work done is seen to depend only on the difference in potential between the two endpoints of the path. If we assume Cartesian coordinates, then Newton's equation $\mathbf{F} = m\mathbf{a}$ becomes

$$m\ddot{\mathbf{x}} = -\nabla V \tag{1.1}$$

when we use the definition of acceleration in terms of position, and write the force in terms of the potential. If we take the dot product of both sides of (1.1) with $\dot{\mathbf{x}}$, we find

$$m\dot{\mathbf{x}} \cdot \ddot{\mathbf{x}} = -\dot{\mathbf{x}} \cdot \nabla V = -\frac{d}{dt} V, \tag{1.2}$$

if we assume for simplicity that the potential V has no explicit time dependence. The result in (1.2) can be rearranged to read

$$\frac{d}{dt}\left(\frac{1}{2} m\, \dot{\mathbf{x}} \cdot \dot{\mathbf{x}} + V\right) = 0. \tag{1.3}$$

The expression in braces in this last result may be identified as the total energy of the particle. The two separate terms are

$$T = \frac{1}{2} m\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}, \tag{1.4}$$

giving the kinetic energy, and V giving the potential energy.

There is another way of formulating Newton's laws which allows for a generalization. Instead of forming the combination $T + V$, which represents the total energy, form

$$L = T - V, \tag{1.5}$$

[3] The motion of the moon does not depend on whether you live in London or in Oshawa, or in how you choose to introduce a coordinate system to describe the motion.

which is called the Lagrangian. L is regarded as a function of the coordinates $\mathbf{x}$ and the velocities $\dot{\mathbf{x}}$, which are treated as independent variables. We have

$$L(\mathbf{x},\dot{\mathbf{x}}) = \frac{1}{2}m\sum_{i=1}^{D}(\dot{x}^i)^2 - V(\mathbf{x}) \tag{1.6}$$

if we use (1.4) with the dot product between the velocities written out explicitly in terms of the components.[4] It is now easy to see that because $\mathbf{x}$ and $\dot{\mathbf{x}}$ are viewed as independent variables, we have

$$\frac{\partial L}{\partial x^i} = -\frac{\partial V}{\partial x^i}$$
$$\frac{\partial L}{\partial \dot{x}^i} = m\dot{x}^i.$$

It therefore follows that

$$0 = \frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) \tag{1.7}$$

for $i = 1,\ldots,D$ are completely equivalent to Newton's equations of motion. We call (1.7) the Euler–Lagrange equations.

It is worth describing what happens if Cartesian coordinates are not used. First of all we would expect that the work done in moving a particle in a given force field should not depend on the choice of coordinates. This is most easily achieved if the potential energy does not depend on the coordinate choice. If we let $\mathbf{q}$ represent any set of coordinates, with $\mathbf{x}$ being reserved for Cartesian coordinates, then we will require

$$V(\mathbf{x}) = \tilde{V}(\mathbf{q}). \tag{1.8}$$

The tilde is necessary in this expression because the functional form of the potential energy may be different for the new and the old coordinates. For example, if $\mathbf{x} = (x,y)$ and we set $\mathbf{q} = (r,\theta)$ with $x = r\cos\theta$ and $y = r\sin\theta$, then for $V(x,y) = x^2 + y^2$ we find $x^2 + y^2 = r^2 = \tilde{V}(r,\theta)$. Clearly, $V(r,\theta) = r^2 + \theta^2 \neq V(x,y)$. If V satisfies (1.8) it is said to be a scalar function.

If we demand that the kinetic energy also be a scalar function of coordinates, then we will be assured that the total energy will be a scalar,

[4] We regard the components of the vector $\mathbf{x}$ in $\mathbb{R}^D$ as x^i with $i = 1,\ldots,D$.

and hence independent of the arbitrary coordinate choice. In Cartesian coordinates we have

$$T = \frac{1}{2}m\sum_{i=1}^{D}(\dot{x}^i)^2 = \frac{1}{2}m\sum_{i=1}^{D}\sum_{j=1}^{D}\delta_{ij}\dot{x}^i\dot{x}^j. \tag{1.9}$$

Here δ_{ij} is the Kronecker delta, defined to be equal to 1 if $i = j$ and equal to 0 if $i \neq j$. We regard $x^i = x^i(\mathbf{q})$, simply expressing the Cartesian coordinates $\mathbf{x}$ in terms of the general coordinates $\mathbf{q}$. The velocity becomes

$$\dot{x}^i = \frac{d}{dt}x^i(\mathbf{q}(t)) = \sum_{j=1}^{D}\frac{\partial x^i}{\partial q^j}\dot{q}^j. \tag{1.10}$$

(We have just used the chain rule for partial differentiation to obtain the last equality here.) Substitution of this expression into the result in (1.9) for the kinetic energy gives

$$T = \frac{1}{2}m\sum_{i=1}^{D}\sum_{j=1}^{D}\delta_{ij}\left(\sum_{k=1}^{D}\frac{\partial x^i}{\partial q^k}\dot{q}^k\right)\left(\sum_{l=1}^{D}\frac{\partial x^j}{\partial q^l}\dot{q}^l\right) = \frac{1}{2}m\sum_{k=1}^{D}\sum_{l=1}^{D}g_{kl}(\mathbf{q})\dot{q}^k\dot{q}^l, \tag{1.11}$$

where we have defined

$$g_{kl}(\mathbf{q}) = \sum_{i=1}^{D}\sum_{j=1}^{D}\delta_{ij}\frac{\partial x^i}{\partial q^k}\frac{\partial x^j}{\partial q^l}. \tag{1.12}$$

$g_{kl}(\mathbf{q})$ is called the metric tensor, and will be familiar to students of differential geometry. (See Laugwitz (1965) for example.)

Due to the proliferation of summation signs, it proves convenient to adopt the Einstein summation convention: any repeated index is summed over the appropriate range of values, in this case $1, 2, \ldots, D$. For example, we would write

$$\dot{x}^i = \sum_{j=1}^{D}\frac{\partial x^i}{\partial q^j}\dot{q}^j$$

as

$$\dot{x}^i = \frac{\partial x^i}{\partial q^j}\dot{q}^j$$

using the Einstein summation convention. The understanding is that because the index j occurs twice in $\left(\partial x^i/\partial q^j\right)\dot{q}^j$, it is summed over all values $1,\ldots,D$.[5] With this new notation we would then rewrite (1.11) and (1.12) as

$$T = \frac{1}{2} m\, g_{kl}(\mathbf{q})\dot{q}^k\dot{q}^l \tag{1.13}$$

and

$$g_{kl}(\mathbf{q}) = \delta_{ij}\,\frac{\partial x^i}{\partial q^k}\,\frac{\partial x^j}{\partial q^l}, \tag{1.14}$$

respectively. Because the repeated indices are arbitrary labels in a summation they can be relabelled at will. This means, for example, that $g_{ij}\dot{q}^i\dot{q}^j$ and $g_{nm}\dot{q}^n\dot{q}^m$ are both equivalent to $g_{kl}\dot{q}^k\dot{q}^l$. The indices of summation are often referred to as dummy indices or dummy labels.

1.2 Hamilton's principle

It is possible to reformulate classical mechanics in order to obtain the Euler–Lagrange equations as a result of a principle of stationary action. To do this we will define the action functional $S[\mathbf{q}(t)]$ as

$$S[\mathbf{q}(t)] = \int_{t_1}^{t_2} dt\, L(q^i(t), \dot{q}^i(t), t). \tag{1.15}$$

Here L is the Lagrangian described in the previous section that, for generality, we allow to have an explicit dependence on time. We allow the use of any coordinate choice here. The use of square brackets in this definition is to symbolize that the action S is a functional, which can be thought of as a function defined on a space of functions.[6] Given some path $\mathbf{q}(t)$, the action is just a real number. If $\mathcal{P}$ denotes the space of all possible paths then we regard $S : \mathcal{P} \to \mathbb{R}$. In contrast to normal functions, the domain of a functional is an infinite dimensional space.[7] Because of the infinite dimensional nature of $\mathcal{P}$, a rigorous treatment of functionals involves some subtle concepts. In particular the notions of differentiation and integration involve some thought. We will proceed in a heuristic manner without full mathematical rigour, as is conventional in physics.

[5] In contrast, the index i only occurs once in each term.

[6] That is the domain of a functional is some space of functions. For the action the domain is the space consisting of all possible particle paths for which the position of the particle is given at the times t_1 and t_2.

[7] Clearly there are an infinite number of paths connecting two points in space.

Hamilton's principle states that the actual motion of a particle whose Lagrangian is L is such that the action functional is stationary (i.e. assumes a maximum or a minimum value). In order to see that this principle is correct we must show that requiring the action to be stationary results in the Euler–Lagrange equations, and conversely that if we impose the Euler–Lagrange equations then the action functional is stationary. The technique for accomplishing this involves the calculus of variations. We will look at what happens to the action functional when the path is varied slightly. We are regarding the endpoints of the path as fixed here, although other choices are possible. Let $\mathbf{q}(t_1) = \mathbf{q}_1$ and $\mathbf{q}(t_2) = \mathbf{q}_2$ be the positions of the particle at the initial time t_1 and the final time t_2 respectively. Let $\mathbf{q}(t)$ be the classical path that the particle follows, and let $\bar{\mathbf{q}}(t)$ be any path with the same two endpoints as the classical path. Now form the difference

$$\delta\mathbf{q}(t) = \bar{\mathbf{q}}(t) - \mathbf{q}(t), \tag{1.16}$$

which represents the deviation of $\bar{\mathbf{q}}(t)$ from the classical path. Because both paths $\bar{\mathbf{q}}(t)$ and $\mathbf{q}(t)$ have the same endpoints, the deviation between the two paths $\delta\mathbf{q}(t)$ must vanish at the initial and final times: $\delta\mathbf{q}(t_1) = \delta\mathbf{q}(t_2) = 0$. The situation is pictured in Fig. 1.1.

We are aiming to show that the action is stationary for the classical path. One way to do this is to assume that the difference $\delta\mathbf{q}(t)$ is small and show that the action is unchanged to first order in the small quantity $\delta\mathbf{q}(t)$. Evaluating the action (1.15) using the arbitrary path $\bar{\mathbf{q}}(t)$ we find

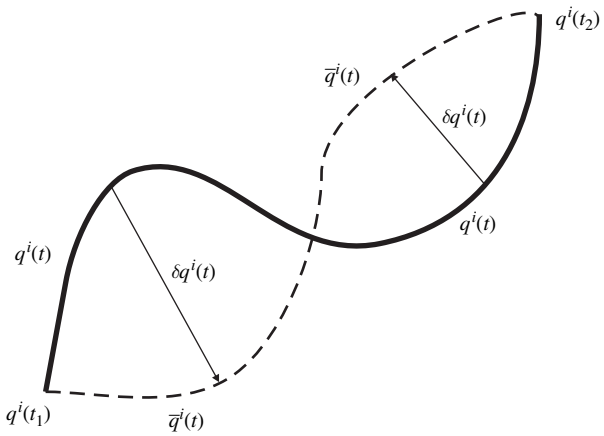

Fig. 1.1 This demonstrates the relationship between the classical path $q^i(t)$ (shown as a solid line), the arbitrary path $\bar{q}^i(t)$ (shown as a dashed line), and the deviation, or difference, $\delta q^i(t)$. The endpoints are held fixed, so that the difference $\delta q^i(t)$ vanishes at times t_1 and t_2.

$$S[\bar{\mathbf{q}}(t)] = \int_{t_1}^{t_2} dt\, L(\bar{q}^i(t), \dot{\bar{q}}^i(t), t)$$
$$= \int_{t_1}^{t_2} dt\, L(q^i(t) + \delta q^i(t), \dot{q}^i(t) + \delta\dot{q}^i(t), t). \tag{1.17}$$

The last equality has just used the definition in (1.16). The Lagrangian is an ordinary function of $q^i(t)$ and $\dot{q}^i(t)$, so may be expanded in an ordinary Taylor series:

$$L\left(q^i(t) + \delta q^i(t), \dot{q}^i(t) + \delta\dot{q}^i(t), t\right) = L\left(q^i(t), \dot{q}^i(t), t\right)$$
$$+\frac{\partial L(q^i(t), \dot{q}^i(t), t)}{\partial q^i(t)}\delta q^i(t) + \frac{\partial L(q^i(t), \dot{q}^i(t), t)}{\partial \dot{q}^i(t)}\delta\dot{q}^i(t) + \cdots . \tag{1.18}$$

Note that the summation convention has been used here, and we have only calculated terms in the expansion up to and including those linear in $\delta q^i(t)$. There will be terms of quadratic and higher order in δq^i in the expansion (1.18), but we will not be concerned with them. Substitution of (1.18) into (1.17) leads to

$$S[\bar{\mathbf{q}}(t)] = S[\mathbf{q}(t)] + \int_{t_1}^{t_2} dt\, \left(\frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial \dot{q}^i}\delta\dot{q}^i\right) + \cdots . \tag{1.19}$$

The second term in the integrand of (1.19) can be rewritten using the identity

$$\frac{\partial L}{\partial \dot{q}^i}\delta\dot{q}^i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\delta q^i\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right)\delta q^i.$$

This results in

$$S[\bar{\mathbf{q}}(t)] = S[\mathbf{q}(t)] + \int_{t_1}^{t_2} dt\, \left[\frac{\partial L}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right)\right]\delta q^i + \frac{\partial L}{\partial \dot{q}^i}\delta q^i\Big|_{t_1}^{t_2} + \cdots ,$$

if we perform the integration over the total time derivative. Because δq^i vanishes at t_1 and t_2, the last term in the line above disappears and we are left with

$$S[\bar{\mathbf{q}}(t)] = S[\mathbf{q}(t)] + \int_{t_1}^{t_2} dt\, \left[\frac{\partial L}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right)\right]\delta q^i, \tag{1.20}$$

to first order in δq^i. This is our main result.

If the Euler–Lagrange equations hold then (1.20) shows that $S[\bar{\mathbf{q}}(t)] = S[\mathbf{q}(t)]$ demonstrating that the action is stationary. Conversely, if

$S[\bar{\mathbf{q}}(t)] = S[\mathbf{q}(t)]$ then we must have the Euler–Lagrange equations holding.[8] This has demonstrated Hamilton's principle: *The action is stationary if and only if the path satisfies the classical equation of motion.* This provides an elegant formulation of classical mechanics with the action functional playing a key role.

1.3 Hamilton's equations

For systems of the type considered in Section 1.1 the Euler–Lagrange equations provide a set of second-order differential equations for the path. With $\mathbf{q} \in \mathbb{R}^D$ we have a set of D coupled equations. It is possible to reformulate these equations as a set of $2D$ coupled first-order differential equations, and this is the essential content of the Hamiltonian form of classical mechanics, along with a physical interpretation of the procedure.

Define the momentum canonically conjugate to the coordinate q^i by

$$p_i = \frac{\partial L}{\partial \dot{q}^i}. \tag{1.21}$$

p_i is usually called just the canonical momentum. Note that for L given by (1.6), $\mathbf{p} = m\dot{\mathbf{x}}$ is the normal definition for the momentum familiar from elementary mechanics. The aim now is to eliminate the dependence on the velocity components $\dot{q}^i$ in favour of the components of canonical momentum p_i. This requires being able to solve the set of equations (1.21) for $\dot{q}^i$ in terms of q^i and p_i. The necessary and sufficient conditions for this are given by the inverse function theorem which states that it is possible to solve (1.21) for $\dot{q}^i$ in terms of q^i and p_i if and only if $\det\left(\partial p_i/\partial \dot{q}^j\right) \neq 0$. A heuristic way to see this is to look at a small variation of (1.21) with q^i held fixed:

$$\delta p_i = \frac{\partial^2 L}{\partial \dot{q}^j \partial \dot{q}^i} \delta \dot{q}^j = \frac{\partial p_i}{\partial \dot{q}^j} \delta \dot{q}^j.$$

It is only possible to solve this for $\delta \dot{q}^j$ if the matrix $\frac{\partial p_i}{\partial \dot{q}^j}$ is invertible. A system which has $\det\left(\partial p_i/\partial \dot{q}^j\right) = 0$ is called singular. We will see examples of singular systems later. For now assume a non-singular system.

Given a non-singular system we can eliminate all dependence on $\dot{q}^i$ by a Legendre transformation:

$$H(\mathbf{q}, \mathbf{p}, t) = p_i \dot{q}^i - L(\mathbf{q}, \dot{\mathbf{q}}, t). \tag{1.22}$$

[8] Formally, if $S[\bar{\mathbf{q}}(t)] = S[\mathbf{q}(t)]$ the integral in (1.20) must vanish for arbitrary $\delta q^i(t)$. We can take $\delta q^i(t)$ to be zero everywhere except for an arbitrary time in the interval $[t_1, t_2]$; thus, the quantity in the integrand of (1.20) appearing in braces must vanish for all times between t_1 and t_2.

H is called the Hamiltonian, and as the notation in (1.22) suggests, it is regarded as a function of the independent variables $\mathbf{q}$ and $\mathbf{p}$, along with a possible time dependence. The Hamiltonian equations are obtained from considering the derivatives of the Hamiltonian with respect to p_i and q^i. To compute the derivative with respect to p_i we use (1.22) and note that apart from the explicit dependence on p_i in the first term, p_i only enters through $\dot{\mathbf{q}}$. This leads to (relabelling the dummy index i in (1.22) to j before differentiating with respect to p_i)

$$\begin{aligned}\frac{\partial H}{\partial p_i} &= \dot{q}^i + p_j \frac{\partial \dot{q}^j}{\partial p_i} - \frac{\partial L}{\partial \dot{q}^j}\frac{\partial \dot{q}^j}{\partial p_i} \\ &= \dot{q}^i, \end{aligned} \tag{1.23}$$

if it is noted that the second and third terms in the first line cancel upon use of (1.21). To compute the derivative of H with respect to q^i we note that $\mathbf{q}$ and $\mathbf{p}$ are the independent variables, and that q^i enters $\dot{q}^j$ through (1.21). This leads to (again relabelling the dummy index i in (1.22) to j)

$$\begin{aligned}\frac{\partial H}{\partial q^i} &= p_j \frac{\partial \dot{q}^j}{\partial q^i} - \frac{\partial L}{\partial q^i} - \frac{\partial L}{\partial \dot{q}^j}\frac{\partial \dot{q}^j}{\partial q^i} \\ &= -\frac{\partial L}{\partial q^i} \quad \text{(by (1.21))} \\ &= -\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) \quad \text{(by (1.7))} \\ &= -\dot{p}_i, \end{aligned} \tag{1.24}$$

where the last line has followed by using (1.21) again. The results contained in (1.23) and (1.24) are called Hamilton's equations. They are seen to consist of a set of first-order differential equations as promised.

Suppose that L takes the standard form

$$L = \frac{1}{2} m\, g_{ij}(\mathbf{q}) \dot{q}^i \dot{q}^j - V(\mathbf{q}),$$

considered in Section 1.1. We can compute p_i from (1.21) to be

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = m\, g_{ij} \dot{q}^j.$$

It is easy to invert this result using the inverse metric g^{ij} defined by

$$g^{ij} g_{jk} = \delta^i{}_k,$$

with $\delta^i{}_k$ the Kronecker delta (i.e. $\delta^i{}_k$ is zero unless the indices i and k are equal, in which case it has the value 1). This gives

$$\dot{q}^i = \frac{1}{m} g^{ij} p_j.$$

The Hamiltonian is now computed from (1.22) to be

$$H = \frac{1}{2m} g^{ij} p_i p_j + V(\mathbf{q}), \tag{1.25}$$

which may be recognized as the sum of the kinetic and potential energies. The Hamiltonian in this case represents the total energy.

In Section 1.2 we showed how the Euler–Lagrange equations could be viewed as a consequence of Hamilton's principle of stationary action. The action functional was defined in (1.15). It is possible to modify the principle of stationary action so that Hamilton's equations (1.23) and (1.24) result. This can be done easily if we note that from (1.22) we have $L = p_i \dot{q}^i - H$. If this is substituted into (1.15) we obtain

$$S[\mathbf{q}, \mathbf{p}] = \int_{t_1}^{t_2} dt \left\{ p_i \dot{q}^i - H(\mathbf{q}, \mathbf{p}, t) \right\}, \tag{1.26}$$

with the action now regarded as a functional of both $\mathbf{q}$ and $\mathbf{p}$. We can now think of the action as a functional of a path in phase space parameterized by the independent coordinates and momenta. To see that Hamilton's equations result from Hamilton's principle of stationary action, perform a variation of (1.26) with independent variations $\delta\mathbf{q}$ and $\delta\mathbf{p}$. This gives

$$\delta S = \int_{t_1}^{t_2} dt \left(\delta p_i \dot{q}^i + p_i \delta \dot{q}^i - \frac{\partial H}{\partial p_i} \delta p_i - \frac{\partial H}{\partial q^i} \delta q^i \right). \tag{1.27}$$

If the second term of (1.27) is integrated by parts, and it is noted that δq^i vanishes at times t_1 and t_2 since the endpoints of the path are held fixed, it is easily seen that

$$\delta S = \int_{t_1}^{t_2} dt \left[\left(\dot{q}^i - \frac{\partial H}{\partial p_i} \right) \delta p_i - \left(\dot{p}_i + \frac{\partial H}{\partial q^i} \right) \delta q^i \right]. \tag{1.28}$$

This result is sufficient to show that Hamilton's equations follow directly from Hamilton's principle of stationary action.

Of course the formalism of Hamiltonian dynamics can be developed much further,[9] and we will consider one line of development in the next

[9] The interested reader should consult Goldstein (1950) or Lanczos (1971) for more details.