

Contents

<i>Preface</i>	<i>page</i> xiii
Introduction	1
The story in a nutshell	4
1 From ordinary to rough differential equations	4
2 Carnot–Caratheodory geometry	8
3 Brownian motion and stochastic analysis	13
I Basics	
1 Continuous paths of bounded variation	19
1.1 Continuous paths on metric spaces	19
1.2 Continuous paths of bounded variation on metric spaces	21
1.3 Continuous paths of bounded variation on \mathbb{R}^d	29
1.4 Sobolev spaces of continuous paths of bounded variation	39
1.5 Comments	44
2 Riemann–Stieltjes integration	45
2.1 Basic Riemann–Stieltjes integration	45
2.2 Continuity properties	49
2.3 Comments	52
3 Ordinary differential equations	53
3.1 Preliminaries	53
3.2 Existence	55
3.3 Uniqueness	59
3.4 A few consequences of uniqueness	60
3.5 Continuity of the solution map	62
3.6 Comments	67
4 ODEs: smoothness	68
4.1 Smoothness of the solution map	68
4.2 Comments	76

5	Variation and Hölder spaces	77
5.1	Hölder and p -variation paths on metric spaces	77
5.2	Approximations in geodesic spaces	88
5.3	Hölder and p -variation paths on \mathbb{R}^d	92
5.4	Generalized variation	99
5.5	Higher-dimensional variation	104
5.6	Comments	111
6	Young integration	112
6.1	Young–Lôeve estimates	112
6.2	Young integrals	115
6.3	Continuity properties of Young integrals	118
6.4	Young–Lôeve–Towghi estimates and 2D Young integrals	119
6.5	Comments	122
II	Abstract theory of rough paths	
7	Free nilpotent groups	125
7.1	Motivation: iterated integrals and higher-order Euler schemes	125
7.2	Step- N signatures and truncated tensor algebras	128
7.3	Lie algebra $\mathfrak{t}^N(\mathbb{R}^d)$ and Lie group $1 + \mathfrak{t}^N(\mathbb{R}^d)$	134
7.4	Chow’s theorem	140
7.5	Free nilpotent groups	142
7.6	The lift of continuous bounded variation paths on \mathbb{R}^d	156
7.7	Comments	163
8	Variation and Hölder spaces on free groups	165
8.1	p -Variation and $1/p$ -Hölder topology	166
8.2	Geodesic approximations	174
8.3	Completeness and non-separability	175
8.4	The d_0/d_∞ estimate	175
8.5	Interpolation and compactness	177
8.6	Closure of lifted smooth paths	178
8.7	Comments	181
9	Geometric rough path spaces	182
9.1	The Lyons-lift map $x \mapsto S_N(x)$	183
9.2	Spaces of geometric rough paths	191
9.3	Invariance under Lipschitz maps	196
9.4	Young pairing of weak geometric rough paths	197
9.5	Comments	211

Contents

ix

10	Rough differential equations	212
10.1	Preliminaries	212
10.2	Davie's estimate	215
10.3	RDE solutions	221
10.4	Full RDE solutions	241
10.5	RDEs under minimal regularity of coefficients	248
10.6	Integration along rough paths	253
10.7	RDEs driven along linear vector fields	262
10.8	Appendix: p -variation estimates via approximations	268
10.9	Comments	279
11	RDEs: smoothness	281
11.1	Smoothness of the Itô–Lyons map	281
11.2	Flows of diffeomorphisms	289
11.3	Application: a class of rough partial differential equations	294
11.4	Comments	301
12	RDEs with drift and other topics	302
12.1	RDEs with drift terms	302
12.2	Application: perturbed driving signals and impact on RDEs	316
12.3	Comments	324
III	Stochastic processes lifted to rough paths	
13	Brownian motion	327
13.1	Brownian motion and Lévy's area	327
13.2	Enhanced Brownian motion	333
13.3	Strong approximations	339
13.4	Weak approximations	354
13.5	Cameron–Martin theorem	357
13.6	Large deviations	359
13.7	Support theorem	367
13.8	Support theorem in conditional form	370
13.9	Appendix: infinite 2-variation of Brownian motion	381
13.10	Comments	383
14	Continuous (semi-)martingales	386
14.1	Enhanced continuous local martingales	386
14.2	The Burkholder–Davis–Gundy inequality	388
14.3	p -Variation rough path regularity of enhanced martingales	390
14.4	Burkholder–Davis–Gundy with p -variation rough path norm	392

x	<i>Contents</i>	
14.5	Convergence of piecewise linear approximations	395
14.6	Comments	401
15	Gaussian processes	402
15.1	Motivation and outlook	402
15.2	One-dimensional Gaussian processes	404
15.3	Multidimensional Gaussian processes	416
15.4	The Young–Wiener integral	433
15.5	Strong approximations	436
15.6	Weak approximations	442
15.7	Large deviations	445
15.8	Support theorem	448
15.9	Appendix: some estimates in $G^3(\mathbb{R}^d)$	451
15.10	Comments	452
16	Markov processes	454
16.1	Motivation	454
16.2	Uniformly subelliptic Dirichlet forms	457
16.3	Heat-kernel estimates	463
16.4	Markovian rough paths	464
16.5	Strong approximations	467
16.6	Weak approximations	480
16.7	Large deviations	483
16.8	Support theorem	484
16.9	Appendix: analysis on free nilpotent groups	493
16.10	Comments	499
IV	Applications to stochastic analysis	
17	Stochastic differential equations and stochastic flows	503
17.1	Working summary on rough paths	503
17.2	Rough paths vs Stratonovich theory	506
17.3	Stochastic differential equations driven by non-semi-martingales	515
17.4	Limit theorems	517
17.5	Stochastic flows of diffeomorphisms	521
17.6	Anticipating stochastic differential equations	523
17.7	A class of stochastic partial differential equations	525
17.8	Comments	526
18	Stochastic Taylor expansions	528
18.1	Azencott-type estimates	528

Contents xi

18.2	Weak remainder estimates	531
18.3	Comments	532
19	Support theorem and large deviations	533
19.1	Support theorem for SDEs driven by Brownian motion	533
19.2	Support theorem for SDEs driven by other stochastic processes	536
19.3	Large deviations for SDEs driven by Brownian motion	538
19.4	Large deviations for SDEs driven by other stochastic processes	541
19.5	Support theorem and large deviations for a class of SPDEs	542
19.6	Comments	544
20	Malliavin calculus for RDEs	545
20.1	\mathcal{H} -regularity of RDE solutions	545
20.2	Non-degenerate Gaussian driving signals	549
20.3	Densities for RDEs under ellipticity conditions	550
20.4	Densities for RDEs under Hörmander's condition	553
20.5	Comments	566

Appendices

A	Sample path regularity and related topics	571
A.1	Continuous processes as random variables	571
A.2	The Garsia–Rodemich–Rumsey estimate	573
A.3	Kolmogorov-type corollaries	582
A.4	Sample path regularity under Gaussian assumptions	587
A.5	Comments	596
B	Banach calculus	597
B.1	Preliminaries	597
B.2	Directional and Fréchet derivatives	598
B.3	Higher-order differentiability	601
B.4	Comments	602
C	Large deviations	603
C.1	Definition and basic properties	603
C.2	Contraction principles	604
D	Gaussian analysis	606
D.1	Preliminaries	606
D.2	Isoperimetry and concentration of measure	608
D.3	L^2 -expansions	610
D.4	Wiener–Itô chaos	610

xii	<i>Contents</i>	
D.5	Malliavin calculus	613
D.6	Comments	614
E	Analysis on local Dirichlet spaces	615
E.1	Quadratic forms	615
E.2	Symmetric Markovian semi-groups and Dirichlet forms	617
E.3	Doubling, Poincaré and quasi-isometry	620
E.4	Parabolic equations and heat-kernels	623
E.5	Symmetric diffusions	625
E.6	Stochastic analysis	627
E.7	Comments	635
	<i>Frequently used notation</i>	636
	<i>References</i>	638
	<i>Index</i>	652