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MULTIDIMENSIONAL STOCHASTIC PROCESSES AS ROUGH PATHS

Rough path analysis provides a fresh perspective on Itô's important theory of stochastic differential equations. Key theorems of modern stochastic analysis (existence and limit theorems for stochastic flows, Freidlin–Wentzell theory, the Stroock–Varadhan support description) can be obtained with dramatic simplifications. Classical approximation results and their limitations (Wong–Zakai, McShane's counterexample) receive "obvious" rough path explanations. Evidence is building that rough paths will play an important role in the future analysis of stochastic partial differential equations, and the authors include some first results in this direction. They also emphasize interactions with other parts of mathematics, including Caratheodory geometry, Dirichlet forms and Malliavin calculus.

Based on successful courses at the graduate level, this up-to-date introduction presents the theory of rough paths and its applications to stochastic analysis. Examples, explanations and exercises make the book accessible to graduate students and researchers from a variety of fields.

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Theory and Applications

PETER K. FRIZ

NICOLAS B. VICTOIR

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To Wendy and Laura

Contents

Preface

This book is split into four parts. **Part I** is concerned with basic material about certain ordinary differential equations, paths of Hölder and variation regularity, and the rudiments of Riemann–Stieltjes and Young integration. Nothing here will be new to specialists, but the material seems rather spread out in the literature and we hope it will prove useful to have it collected in one place.

Part II is about the deterministic core of *rough path theory*, à la T. J. Lyons, but actually inspired by the direct approach of A. M. Davie. Although the theory can be formulated in a Banach setting, we have chosen to remain in a finite-dimensional setting; our motivation for this decision comes from the fact that the bulk of classic texts on Brownian motion and stochastic analysis take place in a similar setting, and these are the grounds on which we sought applications.

In essence, with rough paths one attempts to take out probability from the theory of stochastic differential equations – to the extent possible. Probability still matters, but the problems are shifted from the analysis of the actual SDEs to the analysis of elementary stochastic integrals, known as Lévy's stochastic area. In **Part III** we start with a detailed discussion of how multidimensional Brownian motion can be turned into a (random) rough path; followed by a similar study for (continuous) semi-martingales and large classes of multidimensional Gaussian – and Markovian – processes.

In **Part IV** we apply the theory of rough differential equations (RDEs), path-by-path, with the (rough) sample paths constructed in Part III. In the setting of Brownian motion or semi-martingales, the resulting (random) RDE solutions are identified as solutions to classical stochastic differential equations. We then give a selection of applications to stochastic analysis in which rough path techniques have proved useful.

The prerequisites for Parts I and II are essentially a good command of undergraduate analysis. Some knowledge of ordinary differential equations (existence, uniqueness results) and basic geometry (vector fields, geodesics) would be helpful, although everything we need is discussed. In Part III, we assume a general background in measure theoretic probability theory and the basics of stochastic processes, such as Brownian motion. Stochastic area (for Brownian motion) is introduced via stochastic integration, with alternatives described in the text. In the respective chapters on semi-martingales, Gaussian and Markovian processes, the reader is assumed to have the appropriate background, most of which we have tried to collect in the appendices. Part IV deals with applications to stochastic analysis, stochastic (partial) differential equations in particular. For a full appreciation of the results herein, the reader should be familiar with the relevant background; textbook references are thus given whenever possible at the end of chapters. Exercises are included throughout the text, often with complete (or sketched) solutions.

It is our pleasure to thank our mentors, colleagues and friends. This book would not exist without the teachings of our PhD advisors, S. R. S. Varadhan and T. J. Lyons; both remained available for discussions at various stages throughout the writing process. Once we approached completion, a courageous few offered to do some detailed reading: C. Bayer, M. Caruana, T. Cass, A. Deya, M. Huesmann, H. Oberhauser, J. Teichmann and S. Tindel. Many others offered their time and support in various forms: G. Ben Arous, C. Borrell, F. Baudoin, R. Carmona, D. Chafaï, T. Coulhon, L. Coutin, M. Davis, B. Davies, A. Davie, D. Elworthy, M. Gubinelli, M. Hairer, B. Hambly, A. Iserles, I. Karatzas, A. Lejay, D. Lépingle, P. Malliavin, P. Markowitch, J. Norris, Z. Qian, J. Ramírez, J. Robinson, C. Rogers, M. Sanz-Sole and D. Stroock. This is also a welcome opportunity to thank C. Obtresal, C. Schmeiser, R. Schnabl and W. Wertz for their early teachings. The first author expresses his deep gratitude to the Department of Pure Mathematics and Mathematical Statistics, Cambridge and King's College, Cambridge, where work on this book was carried out under ideal circumstances; he would also like to thank the Radon Institute and his current affiliations, TU and WIAS Berlin, where this book was finalized. Partial support from the Leverhulme Trust and EPSRC grant EP/E048609/1 is gratefully acknowledged. The second author would like to thank the Mathematical Institute, Oxford and Magdalen College, Oxford, where work on the early drafts of this book was undertaken.

Finally, it is our great joy to thank our loving families.

Peter K. Friz (Cambridge, Berlin)
Nicolas B. Victoir (Hong Kong)
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