1

Linear elastic waves

Chapter 1 provides the background, both the model equations and some of the mathematical transformations, needed to understand linear elastic waves. Only the basic equations are summarized, without derivation. Both Fourier and Laplace transforms and their inverses are introduced and important sign conventions settled. The Poisson summation formula is also introduced and used to distinguish between a propagating wave and vibration of a bounded body. A general survey of books and collections of papers that bear on the contents of the book are discussed at the end of the chapter.

A linear wave carries information at a particular velocity, the group velocity, which is characteristic of the propagation structure or environment. It is this transmission of information that gives linear waves their special importance. In order to introduce this aspect of wave propagation, we discuss propagation in one-dimensional periodic structures. Such structures are dispersive and therefore transmit information at a speed different from the wavespeed of their individual components.

1.1 Model equations

The equations of linear elasticity consist of:

(1) the conservation of linear and angular momentum; and
(2) a constitutive relation relating force and deformation.

In the linear approximation the density ρ is constant. The conservation of mechanical energy follows from (1) and (2). The most important feature of the model is that the force exerted across a surface, oriented by the unit normal n_j, by one part of a material on the other is given

by the traction $t_i = \tau_{ji} n_j$, where τ_{ji} is the stress tensor. The conservation of angular momentum makes the stress tensor symmetric: that is, $\tau_{ij} = \tau_{ji}$. The conservation of linear momentum, in differential form, is expressed as

$$\partial_k \tau_{ki} + \rho f_i = \rho \partial_t^2 u_i \,. \tag{1.1.1}$$

The term f_i is a force per unit mass. In general we use Cartesian tensors such as τ_{ij}, where the indices $i, j = 1, 2, 3$, or a bold-face notation $\boldsymbol{\tau}$. The symbol ∂_i is used to represent the partial derivative with respect to the ith coordinate; it is the ith component of the gradient operator ∇. Similarly, sometimes $\mathrm{d}_x f$ is used to represent $\mathrm{d}f/\mathrm{d}x$. Repeated indices are summed over $1, 2, 3$ unless otherwise indicated. For problems engaging only two coordinates, subscripts using Greek letters such as $\alpha, \beta = 1, 2$ are used so that a vector component would be written as u_β and a partial derivative as ∂_α. When these subscripts are repeated they are summed over $1, 2$. At times we use symbols such as c_L or c_T when there is a need to distinguish between parameters that relate to compressional or shear disturbances, but when that distinction is not important we drop the subscript. Constants such as A are used over and over again and have no special meaning.

Deformation is described using a strain tensor

$$\epsilon_{ij} = (\partial_i u_j + \partial_j u_i)/2, \tag{1.1.2}$$

where u_i is ith component of particle displacement. The underlying dependence of the deformation is upon the symmetric part of the displacement gradient $\nabla \mathbf{u}$, which ensures that no rigid body rotations are included. For a homogeneous, isotropic, linearly elastic solid, stress and strain are related by

$$\tau_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}, \tag{1.1.3}$$

where λ and μ are Lamé's elastic constants and δ_{ij} is the Kronecker delta symbol. Substituting (1.1.2) in (1.1.3), followed by substituting the outcome into (1.1.1), gives one form of the equation of motion, namely,

$$(\lambda + \mu)\partial_i \partial_k u_k + \mu \partial_j \partial_j u_i + \rho f_i = \rho \partial_t^2 u_i \,. \tag{1.1.4}$$

Written in vector notation the equation becomes

$$(\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u} + \rho \mathbf{f} = \rho \partial_t^2 \mathbf{u} \,. \tag{1.1.5}$$

Using the identity $\nabla^2\mathbf{u} = \nabla(\nabla\cdot\mathbf{u}) - \nabla\wedge\nabla\wedge\mathbf{u}$, the equation can also be written in the form

$$(\lambda+2\mu)\nabla(\nabla\cdot\mathbf{u}) - \mu\nabla\wedge\nabla\wedge\mathbf{u} + \rho\mathbf{f} = \rho\partial_t^2\mathbf{u}. \tag{1.1.6}$$

This last equation indicates that elastic waves have both dilatational $\nabla\cdot\mathbf{u}$ and rotational $\nabla\wedge\mathbf{u}$ deformations. If $\partial\mathcal{R}$ is the boundary of a region $\mathcal{R}$ occupied by a solid then commonly the tractions $\mathbf{t}$ or displacements $\mathbf{u}$, or combinations of either, are prescribed on $\partial\mathcal{R}$. When $\mathbf{t}$ is given over part of $\partial\mathcal{R}$ and $\mathbf{u}$ over another part, the boundary conditions are said to be *mixed*. One very common boundary condition is to ask that $\mathbf{t} = 0$ everywhere on $\partial\mathcal{R}$. This models the case where a solid surface is adjacent to a gas of such small density and compressibility that it is almost a vacuum. When $\mathcal{R}$ is infinite in one or more dimensions, special conditions are imposed such that a disturbance decays to zero at infinity or radiates outward toward infinity from any sources contained within $\mathcal{R}$.

Another common situation is that in which $\partial\mathcal{R}_{12}$ is the boundary between two regions, 1 and 2, occupied by solids having different properties. Contact between solid bodies is quite complicated, but in many cases it is usual to assume that the traction and displacement, $\mathbf{t}$ and $\mathbf{u}$, are continuous. This models welded contact. One other simple continuity condition that commonly arises is that between a solid and an ideal fluid. Because the viscosity is ignored, the tangential component of $\mathbf{t}$ is set to zero, while the normal component of traction and the normal component of displacement are made continuous.

These are only models and are often inadequate. To briefly indicate some of the possible complications, consider two solid bodies pressed together. A (linear) wave incident on such a boundary would experience continuity of traction and displacement when the solids press together, but would experience a traction-free boundary condition when they pull apart (Comninou and Dundurs, 1977). This produces a complex nonlinear interaction.

The reader may consult Hudson (1980) for a succinct discussion of linear elasticity or Atkin and Fox (1980) for a somewhat more general view.

1.1.1 One-dimensional models

We assume that the various wavefield quantities depend only on the variables x_1 and t. For *longitudinal strain* u_1 is finite, while u_2 and u_3 are assumed to be zero, so that (1.1.2) combined with (1.1.3) becomes

$$\tau_{11} = (\lambda+2\mu)\partial_1 u_1, \quad \tau_{22} = \tau_{33} = \lambda\partial_1 u_1, \tag{1.1.7}$$

and (1.1.1) becomes

$$(\lambda + 2\mu)\partial_1^2 u_1 + \rho f_1 = \rho \partial_t^2 u_1. \tag{1.1.8}$$

For *longitudinal stress* all the stress components except τ_{11} are assumed to be zero. Equation (1.1.3) becomes

$$\tau_{11} = E\partial_1 u_1, \qquad E = \mu \frac{3\lambda + 2\mu}{\lambda + \mu}, \tag{1.1.9}$$

and

$$\partial_2 u_2 = \partial_3 u_3 = -\nu \partial_1 u_1, \qquad \nu = \frac{\lambda}{2(\lambda + \mu)}. \tag{1.1.10}$$

Equation (1.1.1) now becomes

$$E\partial_1^2 u_1 + \rho f_1 = \rho \partial_t^2 u_1 \,. \tag{1.1.11}$$

Equations (1.1.8) and (1.1.11) are essentially the same, though they have somewhat different physical meanings. The longitudinal stress model is useful for rods having a small cross-section and a traction-free surface. Stress components that vanish at the surface are assumed to be negligible in the interior.

1.1.2 Two-dimensional models

We assume that the various wavefield quantities are independent of x_3. As a consequence, (1.1.1) breaks into two separate equations, namely,

$$\partial_\beta \tau_{\beta 3} + \rho f_3 = \rho \partial_t^2 u_3 \tag{1.1.12}$$

and

$$\partial_\beta \tau_{\beta \alpha} + \rho f_\alpha = \rho \partial_t^2 u_\alpha \,. \tag{1.1.13}$$

We use Greek subscripts $\alpha, \beta = 1, 2$ to indicate that the independent spatial variables are x_1 and x_2. The case for which the only non-zero displacement component is $u_3(x_1, x_2, t)$, namely (1.1.12), is called *antiplane shear* motion, or SH motion for shear horizontal. In this case

$$\tau_{3\beta} = \mu \partial_\beta u_3, \tag{1.1.14}$$

giving, from (1.1.12),

$$\mu \partial_\beta \partial_\beta u_3 + \rho f_3 = \rho \partial_t^2 u_3 \,. \tag{1.1.15}$$

Note that this is a two-dimensional scalar equation, similar to (1.1.8) or (1.1.11).

The case for which $u_3 = 0$, while the other two displacement components are generally non-zero, (1.1.13), is called *inplane motion*. The

initials P and SV are used to describe the two types of inplane motion, namely, compressional and shear vertical, respectively. For this case (1.1.3) becomes

$$\tau_{\alpha\beta} = \lambda\partial_\gamma u_\gamma \delta_{\alpha\beta} + \mu(\partial_\alpha u_\beta + \partial_\beta u_\alpha), \tag{1.1.16}$$

and

$$\tau_{33} = \lambda\partial_\gamma u_\gamma. \tag{1.1.17}$$

The equation of motion remains (1.1.4): that is,

$$(\lambda+\mu)\partial_\alpha\partial_\beta u_\beta + \mu\partial_\beta\partial_\beta u_\alpha + \rho f_\alpha = \rho\partial_t^2 u_\alpha. \tag{1.1.18}$$

This last equation is a vector equation and contains two wave types, compressional and shear, whose character we explore shortly. It leads to problems of some complexity.

These two-dimensional equations are the principal models used. The scalar model (1.1.14) allows us to solve complicated problems in detail without being overwhelmed by the size and length of the calculations needed, while the vector model (1.1.18) allows us enough structure to indicate the complexity found in elastic wave propagation.

1.1.3 Displacement potentials

When (1.1.4)–(1.1.6) are used, a boundary condition, such as $\mathbf{t} = 0$, is relatively easy to implement. However, in problems where there are few boundary conditions, it is often easier to cast the equations of motion into a simpler form and allow the boundary condition to become more complicated. One way to do this is to use Helmholtz' theorem (Phillips, 1933, pp. 182–196) to express the particle displacement $\mathbf{u}$ as the sum of a scalar φ and a vector potential $\boldsymbol{\psi}$: that is,

$$\mathbf{u} = \nabla\varphi + \nabla\wedge\boldsymbol{\psi}, \qquad \nabla\cdot\boldsymbol{\psi} = 0. \tag{1.1.19}$$

The second condition is needed because $\mathbf{u}$ has only three components (the particular condition selected is not the only possibility). Assume $\mathbf{f} = 0$. Substituting these expressions into (1.1.6) gives

$$(\lambda+2\mu)\nabla\left(\nabla^2\varphi - \frac{1}{c_L^2}\partial_t^2\varphi\right) + \mu\nabla\wedge\left(\nabla^2\boldsymbol{\psi} - \frac{1}{c_T^2}\partial_t^2\boldsymbol{\psi}\right) = 0. \tag{1.1.20}$$

The equation can be satisfied if

$$c_L^2\nabla^2\varphi = \partial_t^2\varphi, \quad c_L^2 = (\lambda+2\mu)/\rho, \tag{1.1.21}$$

and

$$c_T^2\nabla^2\boldsymbol{\psi} = \partial_t^2\boldsymbol{\psi}, \quad c_T^2 = \mu/\rho. \tag{1.1.22}$$

The terms c_L and c_T are the compressional or longitudinal wavespeed, and shear or transverse wavespeed, respectively. The scalar potential describes a wave of compressional motion, which in the plane-wave case is longitudinal, while the vector potential describes a wave of shear motion, which in the plane-wave case is transverse. Knowing φ and $\boldsymbol{\psi}$, do we know $\mathbf{u}$ completely? Yes we do. Proofs of completeness, along with related references, are given in Achenbach (2003).

1.2 Continuity and boundary conditions

Consider a plane fluid–solid interface oriented by means of a unit normal vector $\hat{\mathbf{n}}$ pointing into the fluid. The traction acting on the surface of the solid is $\mathbf{t_s} = \hat{\mathbf{n}} \cdot \boldsymbol{\tau}$. The continuity conditions at the interface are thus expressed as

$$\mathbf{t}_s \cdot \hat{\mathbf{n}} = -p_f, \quad \hat{\mathbf{n}} \wedge \mathbf{t}_s = 0, \quad \mathbf{u}_s \cdot \hat{\mathbf{n}} = \mathbf{u}_f \cdot \hat{\mathbf{n}}. \tag{1.2.1}$$

Because the fluid is ideal, no condition is placed on $\hat{\mathbf{n}} \wedge \mathbf{u}_{f,s}$.

The only other boundary condition needed in this work is one at infinity. The waves must in general be outgoing, though when the focused beam is discussed an incoming wave is considered. The *principle of limiting absorption* (Harris, 2001, pp. 62, 63) is used, in most cases, to determine this. Either by Fourier transforming a signal or by considering a time-harmonic one, in the far field, it will have the form

$$\varphi = \frac{A(\phi, \theta)}{kr} \, \mathrm{e}^{\mathrm{i}(kr - \omega t)},$$

where (r, ϕ, θ) are spherical coordinates, and $k = \omega/c$ is the wavenumber, with c being the wavespeed. The angular frequency is defined such that $\omega = \omega_0 + \mathrm{i}\epsilon$, $\omega_0 > 0$, $\epsilon \geq 0$. The wavenumber then becomes $k = (\omega_0/c) + \mathrm{i}(\epsilon/c)$. Therefore

$$|\varphi| \sim \mathrm{e}^{-\epsilon r/c} \, \mathrm{e}^{\epsilon t}, \tag{1.2.2}$$

as $r \to \infty$ with t fixed; that is, the wave vanishes provided the combination $[\mathrm{i}(kr - \omega t)]$ appears in some guise. The parameter ϵ can be sent to zero at the end of the calculations.

1.3 Flux of energy

The remaining conservation law of importance is the conservation of mechanical energy. Again assume $\mathbf{f} = 0$. This law can be derived directly

from (1.1.1)–(1.1.3) by taking the dot product of $\partial_t \mathbf{u}$ with (1.1.1). This gives, initially,

$$(\partial_j \tau_{ji})\partial_t u_i - \rho(\partial_t^2 u_i)\partial_t u_i = 0. \tag{1.3.1}$$

Forming the product $\tau_{kl}\epsilon_{kl}$, using (1.1.3) and making use of the decomposition $\partial_j u_i = \epsilon_{ji} + \omega_{ji}$, where $\omega_{ji} = (\partial_j u_i - \partial_i u_j)/2$, allows us to write (1.3.1) as

$$\frac{1}{2}\partial_t \left\{\rho(\partial_t u_i)(\partial_t u_i) + \tau_{ki}\epsilon_{ki}\right\} + \partial_k \left(-\tau_{ki}\partial_t u_i\right) = 0. \tag{1.3.2}$$

The first two terms on the left-hand side become the time rates of change of

$$\mathcal{K} = \frac{1}{2}\rho(\partial_t u_k)(\partial_t u_k), \qquad \mathcal{U} = \frac{1}{2}\tau_{ij}\epsilon_{ij}. \tag{1.3.3}$$

These are the kinetic and internal energy density, respectively. The remaining term is the divergence of the energy flux, $\boldsymbol{\mathcal{F}}$, where $\mathcal{F}_j$ is given by

$$\mathcal{F}_j = -\tau_{ji}\partial_t u_i \,. \tag{1.3.4}$$

Equation (1.3.2) can then be written as

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \boldsymbol{\mathcal{F}} = 0, \tag{1.3.5}$$

where $\mathcal{E} = \mathcal{K} + \mathcal{U}$ and is the energy density. This is the differential statement of the conservation of mechanical energy. To better understand that the energy flux or power density is given by (1.3.4), consider an arbitrary region $\mathcal{R}$, with surface $\partial\mathcal{R}$. Integrating (1.3.5) over $\mathcal{R}$ and using Gauss' theorem gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{R}} \mathcal{E}(\mathbf{x}, t)\,\mathrm{d}V = -\int_{\partial\mathcal{R}} \boldsymbol{\mathcal{F}} \cdot \hat{\mathbf{n}}\,\mathrm{d}S. \tag{1.3.6}$$

Therefore, as the mechanical energy decreases within $\mathcal{R}$, it radiates outward across the surface $\partial\mathcal{R}$ at a rate $\boldsymbol{\mathcal{F}} \cdot \hat{\mathbf{n}}$.

Because time-harmonic problems are being considered (see the next section), the time average of the flux of energy per unit area, or the intensity, is of interest. The time average of a quantity $a(\mathbf{x}, t)$ is defined as

$$\langle a(\mathbf{x}, t)\rangle := \frac{1}{T}\int_t^{t+T} a(\mathbf{x}, \tau)\mathrm{d}\tau, \tag{1.3.7}$$

where $T = 2\pi/\omega$, and the time-dependence is $\mathrm{e}^{-\mathrm{i}\omega t}$. Given two terms

$$a(\mathbf{x}, t) = \mathrm{Re}[a(\mathbf{x})\mathrm{e}^{-\mathrm{i}\omega t}], \quad b(\mathbf{x}, t) = \mathrm{Re}[b(\mathbf{x})\mathrm{e}^{-\mathrm{i}\omega t}],$$

the time average of their product is

$$\langle a(\mathbf{x},t)\; b(\mathbf{x},t)\rangle = \frac{1}{2}\mathrm{Re}[a(\mathbf{x})\, b^*(\mathbf{x})]. \tag{1.3.8}$$

This is derived by a direct substitution of the product into (1.3.7). The superscript * indicates the complex conjugate.

Equation (1.3.8) is especially useful when calculating the time-average flux of energy per unit area, which is also called the intensity. Expressing the stress tensor and particle displacement as

$$\boldsymbol{\tau}(\mathbf{x},t) = \mathrm{Re}[\boldsymbol{\tau}(\mathbf{x})\mathrm{e}^{-\mathrm{i}\omega t}], \quad \mathbf{u}(\mathbf{x},t) = \mathrm{Re}[\mathbf{u}(\mathbf{x})\mathrm{e}^{-\mathrm{i}\omega t}],$$

the time average of (1.3.4) is

$$\langle \boldsymbol{\mathcal{F}}\rangle = \frac{1}{2}\mathrm{Im}[\omega\, \boldsymbol{\tau}(\mathbf{x})\cdot\mathbf{u}^*(\mathbf{x})]. \tag{1.3.9}$$

Two additional time-average quantities will be of interest: the time-average flux of energy $\langle\mathcal{P}\rangle$ across a surface $\partial\mathcal{S}$ oriented by the unit normal $\hat{\mathbf{n}}$, and its complex counterpart $\langle\mathcal{P}\rangle^c$. These are given by

$$\langle\mathcal{P}\rangle = \mathrm{Re}\langle\mathcal{P}\rangle^c = \int_{\partial\mathcal{S}} \langle\boldsymbol{\mathcal{F}}\rangle\cdot\hat{\mathbf{n}}\,\mathrm{d}S \tag{1.3.10}$$

and

$$\langle\mathcal{P}\rangle^c = \frac{\mathrm{i}\omega}{2}\int_{\partial\mathcal{S}} (\boldsymbol{\tau}\cdot\mathbf{u}^*)\cdot\hat{\mathbf{n}}\,\mathrm{d}S. \tag{1.3.11}$$

Lastly, in all the cases treated in this book it can be shown (Auld, 1990a, pp. 201–207; Lighthill, 1965) that

$$\langle\boldsymbol{\mathcal{F}}\rangle = \boldsymbol{\mathcal{C}}\langle\mathcal{E}\rangle, \tag{1.3.12}$$

where $\boldsymbol{\mathcal{C}}$ is the group velocity and the energy density $\mathcal{E}$ is given following (1.3.5).

Cautionary note. There are waves, such as mode L_3 of an elastic plate (see §3.2 and §8.2), whose group velocity $\boldsymbol{\mathcal{C}}$ and wave (or phase) velocity $\mathbf{c}$ are in different directions. In such cases, the principle of limiting absorption, (1.4.2), must be applied with care, and it is more direct to ask that $\langle\boldsymbol{\mathcal{F}}\rangle$, or equivalently $\boldsymbol{\mathcal{C}}$, be directed away from the source.

1.4 The Fourier and Laplace transforms

All waves are transient in time. One useful representation of a transient waveform is its Fourier one. This representation imagines the transient signal decomposed into an infinite number of time-harmonic or frequency components. One important reason for the usefulness of this representation is that the transmitter, receiver, and propagation structure usually respond differently to the different frequency components. The linearity of the problem ensures that we can work out the net propagation outcomes for each frequency component and then combine the outcomes to recreate the received signal.

The Fourier transform is defined as

$$\bar{u}(x,\omega) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega t} u(x,t)\mathrm{d}t. \tag{1.4.1}$$

The variable ω is complex. Its domain is such as to make the above integral convergent. For $t > 0$ this domain is $\mathrm{Im}(\omega) > 0$. $\bar{u}$ is an analytic function within the domain of convergence, though it can be analytically continued beyond it.[1] The inverse transform is defined as

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega t} \bar{u}(x,\omega)\mathrm{d}\omega\,. \tag{1.4.2}$$

Thus we have represented u as a sum of harmonic waves $\mathrm{e}^{-\mathrm{i}\omega t}\bar{u}(x,\omega)$. *Note that there is a specific sign convention for the exponential term that we shall adhere to throughout the book.*

A closely related transform is the Laplace one. It is usually used for initial-value problems so that we imagine that, for $t < 0$, $u(x,t)$ is zero. This is not essential and its definition can be extended to include functions that extend through values of negative t. This transform is defined as

$$\tilde{u}(x,p) = \int_{0}^{\infty} \mathrm{e}^{-pt} u(x,t)\mathrm{d}t. \tag{1.4.3}$$

As with ω, p is a complex variable and its domain is such as to make $\tilde{u}(x,p)$ an analytic function of p. The domain is initially defined as $\mathrm{Re}(p) > 0$, but the function can be analytically continued beyond this. Note that $p = -\mathrm{i}\omega$ so that $\mathrm{Im}(\omega) > 0$. The inverse transform is given by

$$u(x,t) = \frac{1}{2\pi\mathrm{i}} \int_{\epsilon-\mathrm{i}\infty}^{\epsilon+\mathrm{i}\infty} \mathrm{e}^{pt} \tilde{u}(x,p)\mathrm{d}p, \tag{1.4.4}$$

[1] Analytic functions defined by contour integrals, including the case in which the contour extends to infinity, are discussed in Titchmarsh (1939, pp. 85–86) in a general way and in more detail by Noble (1988).

where $\epsilon \geq 0$. The expressions for the inverse transforms (1.4.2) and (1.4.4) are misleading. In practice we define the inverse transforms on contours that are designed to capture the physical features of the solution. A large part of this book will deal with just how those contours are selected. But, for the present, we shall work with these definitions.

Consider the case of longitudinal strain. Imagine that at $t = -\infty$ a disturbance started with zero amplitude. Taking the Fourier transform of (1.1.8) gives

$$\frac{\mathrm{d}^2\bar{u}_1}{\mathrm{d}x_1^2} + k^2\bar{u}_1 = 0, \tag{1.4.5}$$

where $k = \omega/c_L$ and c_L is the compressional wavespeed defined in (1.1.21). The parameter k is called the wavenumber. Equation (1.4.5) has solutions of the form

$$\bar{u}_1(x_1, \omega) = A(\omega)\mathrm{e}^{\pm \mathrm{i}kx_1}. \tag{1.4.6}$$

If we had sought a time-harmonic solution of the form

$$u_1(x_1, t) = \bar{u}_1(x_1, \omega)\mathrm{e}^{-\mathrm{i}\omega t}, \tag{1.4.7}$$

we should have gotten the same answer except that $\mathrm{e}^{-\mathrm{i}\omega t}$ would be present. In other words, taking the Fourier transform of an equation over time or seeking solutions that are time-harmonic are two slightly different ways of doing the same operation.

For (1.4.7), it is understood that the real part can always be taken to obtain a real disturbance. Much the same happens in using (1.4.2). In writing (1.4.2) we implicitly assumed that $u(x, t)$ was real. That being the case, $\bar{u}(x, \omega) = \bar{u}^*(x, -\omega)$, where the superscript * to the right of the symbol indicates the complex conjugate. From this it follows that

$$u(x, t) = \frac{1}{\pi}\,\mathrm{Re}\int_0^\infty \mathrm{e}^{-\mathrm{i}\omega t}\bar{u}(x, \omega)\mathrm{d}\omega\,. \tag{1.4.8}$$

The advantage of this formulation of the inverse transform is that we may proceed with all our calculations using an implied $\mathrm{e}^{-\mathrm{i}\omega t}$ and assuming ω is positive. The importance of this will become apparent in subsequent chapters. Equation (1.4.8) can be regarded as a generalization of taking the real part of a time-harmonic wave (1.4.7).

Equation (1.4.6), when the + sign is taken, is a time-harmonic, plane wave propagating in the positive x_1 direction. *We assume that the wavenumber k is positive, unless otherwise stated.* The wave propagates in the positive x_1 direction because the term $(kx_1 - \omega t)$ remains constant, and hence u_1 remains constant, only if x_1 increases as t increases. The