1
Introduction

1.1 Welcome

This book reports on the state of a research program that I initiated in 1999. It connects the practice of proper forcing introduced by Shelah [64] with the study of various σ-ideals on Polish spaces from the point of view of abstract analysis, descriptive set theory, measure theory, etc. It turns out that the connection is far richer than I dared to imagine in the beginning. Its benefits include theorems about methodology of forcing as well as isolation of new concepts in measure theory or abstract analysis. It is my sincere hope that this presentation will help to draw attention from experts from these fields and to bring set theory and forcing closer to the more traditional parts of mathematics.

The book uses several theorems and proofs from my earlier papers; in several cases I coauthored these papers with others. The first treatment of the subject in [83] is superseded here on many accounts, but several basic theorems and proofs remain unchanged. The papers [18], [67], [82], [86], and [87] are incorporated into the text, in all cases reorganized and with significant improvements.

Many mathematicians helped to make this book what it is. Thanks should go in the first place to Bohuslav Balcar for his patient listening and enlightening perspective of the subject. Vladimir Kanovei introduced me to effective descriptive set theory. Ilijas Farah helped me with many discussions on measure theory. Joerg Brendle and Peter Koepke allowed me to present the subject matter in several courses, and that greatly helped organize my thoughts and results. Last but not least, the influence of the mathematicians I consider my teachers (Thomas Jech, Hugh Woodin, and Alexander Kechris) is certainly apparent in the text.

I enjoyed financial support through NSF grant DMS 0300201 and grant GA ČR 201-03-0933 of the Grant Agency of Czech Republic as I wrote this book.

1.2 Navigation

This is not a textbook. The complexity of the subject is such that it is impossible to avoid forward references and multiple statements of closely related results, and to keep the book organized in a logical structure at the same time. As a result, the linear reading of the book will be necessarily interspersed with some page flipping. This section should help the reader to find the subjects he is most interested in.

Chapter 2 provides the basic definitions, restatements of properness, and basic implications of properness, such as the reading of reals in the generic extension as images of the generic point under ground model coded Borel functions. Every reader should start with this chapter. A sample theorem:

Theorem 1.2.1. *Suppose that I is a σ-ideal on a Polish space X. The forcing P_I of I-positive Borel sets ordered by inclusion adds a single point $\dot{x}_{gen} \in X$ such that a set B belongs to the generic filter if and only if it contains the generic point $\dot{x}_{gen}$.*

Chapter 3 investigates the possible finer forcing properties of the forcings of the form P_I. These divide into three basic groups. The first group is that of Fubini forcing properties, introduced in Section 3.2. These correspond to the classical preservation properties such as the bounding property or preservation of outer Lebesgue measure. A sample theorem:

Theorem 1.2.2. *Suppose that I is a σ-ideal on a Polish space X such that the forcing P_I is proper. The following are equivalent:*

1. *P_I is bounding;*
2. *for every Polish topology τ on the space X that yields the same Borel structure as the original one, every Borel I-positive set contains a τ-compact I-positive subset.*

The second group of properties is entirely absent in the combinatorial treatment of forcings. These are the descriptive set theoretic properties of the ideals, represented by the various dichotomies of Section 3.9 and the $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ property. The dichotomies are constantly invoked in the proofs of absoluteness theorems and preservation theorems. The $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ property of ideals allows ZFC treatment of such operations as the countable support iteration, product, and illfounded iteration, with a more definite understanding of the underlying issues. A sample theorem:

Theorem 1.2.3. *(LC+CH) Suppose that I is a σ-ideal generated by a universally Baire collection of analytic sets such that every I-positive $\mathbf{\Sigma}^1_2$ set has an I-positive Borel subset. If the forcing P_I is ω-proper then every function $f \in 2^{\omega_1}$ in the extension either is in the ground model or has a countable initial segment which is not in the ground model.*

Here, LC denotes a suitable large cardinal assumptions, as explained in the next section.

The third group of properties is connected with determinacy of games on Boolean algebras. A number of forcing properties can be expressed in terms of infinitary games of the poset P_I which are determined in the definable context. The games are usually variations on standard fusion arguments, and the winning strategies are a necessary tool in the treatment of product forcing, illfounded iteration, and other subjects. A sample application:

Theorem 1.2.4. *(LC) Suppose that I is a universally Baire σ-ideal on a Polish space X such that the forcing P_I is proper. The following are equivalent:*

1. *P_I preserves Baire category;*
2. *there is a collection T of Polish topologies on the space X such that I is the collection of all sets which are τ-meager for every topology $\tau \in T$.*

Chapter 4 gives a number of classes of σ-ideals I for which I can prove that the forcing P_I is proper. While the presentation is based on a joint paper with Ilijas Farah [18], it is nevertheless greatly expanded. There are two very distinct groups of ideals in this respect: the ideals satisfying the first dichotomy, whose treatment occupies almost the whole chapter, and the ideals that do not satisfy the first dichotomy, treated in Section 4.7. It seems that the former group is much larger. Its treatment is divided into several very populous subgroups, each treated in its own section. These subgroups are typically connected with a basic underlying idea from abstract analysis, such as capacities or Hausdorff measures. The sections are all very much alike: first comes the definition of the class of ideals, then the properness theorem, then the dichotomy theorem (which, mysteriously, is always proved in the same way as properness), then several general theorems regarding the finer forcing properties of the ideals. The section closes with a list of examples. A sample result:

Theorem 1.2.5. *Suppose that ϕ is an outer regular subadditive capacity on a Polish space X. Let $I = \{A \subset X : \phi(A) = 0\}$. Then:*

1. *if the capacity is stable then the forcing P_I is proper;*
2. *if the forcing P_I is proper and the capacity is strongly subadditive then the forcing P_I preserves outer Lebesgue measure;*
3. *if the forcing P_I is proper and the capacity is Ramsey then the forcing does not add splitting reals;*
4. *every capacity used in potential theory is stable.*

My original hope that the idealization of forcings would closely relate to the creature forcing technology [58] proved to be naive; the symmetric difference of the two approaches turned out to be quite large. Nevertheless, in several cases I could identify a precise correspondence between a class of ideals and a class of creature forcings.

Chapter 5 relates operations on ideals with operations on forcings. The key case here is that of the countable support iteration which corresponds to a transfinite Fubini product of ideals, Section 5.1. The other operations I can handle are side-by-side product with a great help from determinacy of games on Boolean algebras, the illfounded iteration, which provides a treatment dual to and more general than that of [43], the towers of ideals which is a method of obtaining forcings adding objects more complex than just reals, and the union of ideals, which forcingwise is an entirely mysterious operation. A sample theorem:

Theorem 1.2.6. *(LC) Suppose that $I_\alpha : \alpha \in \kappa$ is a collection of universally Baire σ-ideals on some Polish spaces such that the forcings P_{I_α} are all proper and preserve Baire category bases. Then the countable support side-by-side product of these forcings is proper as well and preserves Baire category bases. In addition, the ideals satisfy a rectangular Ramsey property.*

Chapter 6 is probably the primary reason why a forcing practitioner may want to read this book; however its methods are entirely incomprehensible without the reading of the previous chapters. There are several separate sections.

Section 6.1 contains the absoluteness results which originally motivated the work on the subject of this book. There are many theorems varying in the exact large cardinal strength necessary and in the class of problems they can handle, but on the heuristic level they all say the same thing. If $\mathfrak{x}$ is a simply definable cardinal invariant and I is a σ-ideal such that the forcing P_I is proper, then if the inequality $\mathfrak{x} < \mathtt{cov}^*(I)$ holds in some extension then it holds in the iterated P_I extension. Moreover, there is a forcing axiom CPA(I) which holds in the iterated P_I extension and which then must directly imply the inequality $\mathfrak{x} < \mathtt{cov}^*(I)$. The CPA-type axioms have been defined independently in the work of Ciesielski and Pawlikowski [9] in an effort to axiomatize the iterated Sacks model. A sample theorem:

Theorem 1.2.7. *(LC) Suppose that $\mathfrak{x}$ is a tame cardinal invariant and $\mathfrak{x} < \mathfrak{c}$ holds in some forcing extension. Then $\aleph_1 = \mathfrak{x} < \mathfrak{c}$ holds in every forcing extension satisfying CPA; in particular it holds in the iterated Sacks model.*

Section 6.2 considers the duality theorems. These are theorems that partially confirm the old duality heuristic: if I, J are σ-ideals and the inequality $\mathtt{cov}(I) \leq \mathtt{add}(J)$ is provable in ZFC, then so should be its dual inequality $\mathtt{non}(I) \geq \mathtt{cof}(J)$. This is really completely false, but several theorems can be proved that rescue nontrivial pieces of this unrealistic expectation. This is the one part of this book where the combinatorics of uncountable cardinals actually enters the computation of inequalities between cardinal invariants, with considerations involving various pcf and club guessing structures. A sample theorem:

Theorem 1.2.8. *Suppose that J is a σ-ideal on a Polish space generated by a universally Baire collection of analytic sets. If ZFC+LC proves* $\mathtt{cov}(I) = \mathfrak{c}$ *then ZFC + LC proves* $\mathtt{non}(I) \leq \aleph_2$.

Section 6.3 gives a long list of preservation theorems for the countable support iteration of definable forcings. Compared to the combinatorial approach of Shelah [64], these theorems have several advantages: they connect well with the motivating problems in abstract analysis, and they have an optimal statement. Among their disadvantages I must mention the restriction to definable forcings and the necessity of large cardinal assumptions for a full strength version. Many of the preservation theorems of this section have no combinatorial counterpart. A sample result:

Theorem 1.2.9. *(LC) Suppose that I is a universally Baire σ-ideal on a Polish space X such that the forcing P_I is proper. Suppose that ϕ is a strongly subadditive capacity. If P_I forces every set to have the same ϕ-mass in the ground model as it has in the extension, then even the countable support iterations of the forcing P_I have the same property.*

1.3 Notation

My notation follows the set theoretic standard of [29]. If T is a tree of finite sequences ordered by extension then $[T]$ denotes the set of all infinite paths through that tree; if $T \subset 2^{<\omega}$ then $[T]$ is a closed subset of the space 2^ω. If X, Y are Polish spaces and $A \subset X \times Y$ is a set then the expression $\mathrm{proj}(A)$ denotes the set $\{x \in X : \exists y \in Y \ \langle x, y\rangle \in A\}$, for a point $x \in X$ the expression A_x stands for the vertical section $\{y \in Y : \langle x, y\rangle \in A\}$, and for a point $y \in Y$ the expression A^y stands for the horizontal section $\{x \in X : \langle x, y\rangle \in A\}$. For a Polish space X, $K(X)$ is the hyperspace of its compact subsets with the Vietoris topology and $P(X)$ is the space of probability Borel measures on X. The expression $\mathcal{B}(X)$ denotes the collection of all Borel subsets of the space X. The word "measure" refers to a σ-additive Borel measure. If a set function is σ-subadditive rather than σ-additive then I use the word "submeasure." The value of a measure (submeasure, capacity) ϕ at a set B is referred to as the ϕ-mass of the set B. A tower of models is a sequence $\langle M_\alpha : \alpha \in \beta\rangle$ where β is an ordinal and M_α's are elementary submodels of some large structure (typically $\langle H_\theta, \in\rangle$ for a suitable large cardinal θ) such that

$\alpha' \in \alpha \in \beta$ implies $M_{\alpha'} \in M_\alpha$. The tower is continuous if for limit ordinals $\alpha \in \beta$, $M_\alpha = \bigcup_{\gamma \in \alpha} M_\gamma$.

One important deviation from the standard set theoretical usage is the liberal use of large cardinal assumptions. In order to prove suitably general theorems of a statement that is easy to understand and refer to, I frequently have to resort to a large cardinal assumption of this or that kind. There are only three classes of applications of large cardinal assumptions in this book–absoluteness, determinacy of (long and complex) games, and definable uniformization. The minimum large cardinal necessary for each of these applications is different, sometimes difficult to state, sometimes unknown, and invariably completely irrelevant for the goals of this book; the existence of a supercompact cardinal is always sufficient. As a result, I decided to denote the use of large cardinal assumptions by a simple (LC) preceding the statement of the theorems. For most but not all *specific* applications of the general theorems in this book the large cardinal assumption can be eliminated by manual construction of all the winning strategies and uniformization functions necessary. At least in one case (the countable support iteration of Laver forcing) I made an effort to show that the key dichotomy requires a large cardinal assumption, and in the rather restrictive case of $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ ideals almost all general theorems in this book are proved in ZFC.

The labeling of the various claims in this book is indicative of their position and function. Facts are statements that are proved elsewhere, and I will not restate their proofs. Theorems are quotable self-standing statements, ready for use in the reader's work. Propositions are self-standing statements referred to at some other, possibly quite distant, place in the book. Finally, claims and lemmas appear in the proofs of theorems and propositions, and they are not referred to in any other place.

1.4 Background

The subject of this book demands the reader to be proficient in several areas of set theory and willing to ask at least the basic questions about several other fields of mathematics. This section sums up the basic definitions and results which are taken for granted in the text.

1.4.1 Polish spaces

A *Polish space* is a separable completely metrizable topological space. Many Polish spaces occur in this book. If T is a countably branching tree without endnodes, then the set $[T]$ of all infinite branches through the tree T equipped with the topology generated by the sets $O_t = \{x \in [T] : t \subset x\}$ is a Polish space, with important special cases the Cantor space 2^ω and the Baire space ω^ω.

I will make use of basic theory of Polish spaces as exposed in [40]. Every uncountable Polish space X is a Borel bijective image of the Cantor space and it is a continuous bijective image of a closed subset of the Baire space. A G_δ subset of a Polish space is again Polish in the inherited topology. Every Polish space is homeomorphic to a G_δ subset of the Hilbert cube.

There are several useful operations on Polish spaces. If X, Y are Polish spaces then their product is again Polish; even a product of countably many Polish spaces is still Polish. If X is a Polish space then $K(X)$ denotes the space of all compact subsets of X equipped with *Vietoris topology* generated by sets of the form $\{K \in K(X) : K \subset O\}$ and $\{K \in K(X) : K \cap O \neq 0\}$ for open sets $O \subset X$. The space $K(X)$ is referred to as the *hyperspace* of X; it is Polish and if X is compact then $K(X)$ is compact as well.

It is possible to change the topology on a Polish space to a new, more convenient one. Whenever X is Polish with topology τ and $B_n : n \in \omega$ are τ-Borel subsets of X then there is a Polish topology η extending τ such that the sets $B_n : n \in \omega$ are η-clopen and the η-Borel sets are exactly the τ-Borel sets.

1.4.2 Definable subsets of Polish spaces

Definability of subsets of Polish spaces plays a critical role. Let X be a Polish space, with a countable topology basis $\mathcal{O}$. *Borel sets* are those sets which can be obtained from the basic open sets by a repeated application of countable union, countable intersection, and taking a complement. This is a class of sets closed under continuous preimages and continuous one-to-one images, but not under arbitrary continuous images. *Analytic sets* are those that can be obtained as continuous images of Borel sets. This is a class of sets containing the Borel sets, closed under continuous images, countable unions and intersections, but not under complements. Every analytic set $A \subset X$ is a projection of a closed subset $C \subset X \times \omega^\omega$, $A = \text{proj}(C)$. Every analytic subset of the Baire space is of the form $\text{proj}[T]$. Every analytic set whose complement is analytic is in fact Borel.

The paper [20] isolated an important and very practical broad definability class of subsets of Polish spaces. A set $A \subset 2^\omega$ is *universally Baire* if there are class trees $S, T \subset (2 \times \text{Ord})^{<\omega}$ which in all set generic extensions project into complementary subsets of 2^ω and $A = \text{proj}[T]$. A subset of another Polish space is universally Baire if it is in Borel bijective correspondence with a universally Baire subset of the Cantor space. Equivalently, a set is universally Baire if all of its continuous preimages have the property of Baire.

In ZFC, analytic sets and coanalytic sets are universally Baire, and consistently the class of universally Baire sets does not reach far beyond that. However, under large cardinal assumptions the class of universally Baire sets expands considerably. If there is a proper class of Woodin cardinals then the class of universally Baire sets is closed under complementation and continuous images and preimages, and every set of reals in the model $L(\mathbb{R})$ is universally Baire.

1.4.3 Measure theory

Let X be a Polish space. A *submeasure* on X is a map $\phi : \mathcal{P}(X) \to \mathbb{R}^+$ such that $\phi(0) = 0$, $A \subset B \to \phi(A) \leq \phi(B)$ and $\phi(\bigcup_n A_n) \leq \Sigma_n \phi(A_n)$ whenever $A_n : n \in \omega$ is a countable collection of subsets of the space X. The submeasures on uncountable Polish spaces in this book will always be countably subadditive in this sense. The submeasure ϕ is *outer regular* if $\phi(A) = \inf\{\phi(O) : A \subset O, O \text{ open}\}$ and it is *outer* if $\phi(A) = \inf\{\phi(B) : A \subset B : B \text{ Borel}\}$.

A Borel measure (or *measure*) is a map $\phi : \mathcal{B}(X) \to \mathbb{R}^+$ such that $\phi(0) = 0$, $A \subset B \to \phi(A) \leq \phi(B)$ and $\phi(\bigcup_n A_n) = \Sigma_n \phi(A_n)$ if $A_n : n \in \omega$ is a countable collection of pairwise disjoint Borel sets. Finite Borel measures on Polish spaces are outer regular and tight: $\phi(A) = \inf\{\phi(O) : A \subset O, O \text{ open}\} = \sup\{\phi(K) : K \subset A, K \text{ compact}\}$. I will need a criterion for the restriction of a submeasure ϕ on X to the Borel subsets of X to be a measure. If d is a complete separable metric on X and for every pair of closed sets $C_0, C_1 \subset X$ which are nonzero distance apart, $\phi(C_0 \cup C_1) = \phi(C_0) + \phi(C_1)$ then indeed $\phi \restriction \mathcal{B}(X)$ is a measure. In this situation I will say that ϕ is a *metric measure*.

A capacity on a Polish space X is a map $\phi : \mathcal{P}(X) \to \mathbb{R}^+$ such that $\phi(0) = 0$, $A \subset B \to \phi(A) \leq \phi(B)$, $\phi(\bigcup_n A_n) = \sup_n \phi(A_n)$ whenever $A_n : n \in \omega$ is a countable inclusion-increasing sequence of subsets of the space X, and $\phi(K) = \inf\{\phi(O) : K \subset O, O \text{ open}\}$ for compact sets $K \subset X$. Capacities are tight on analytic sets: if $A \subset X$ is analytic then $\phi(A) = \sup\{\phi(K) : K \subset A : K \text{ compact}\}$.

1.4.4 Determinacy

Infinitary games of all kinds, lengths, and complexities are a basic feature of this book. The key problem always is whether one of the players must have a winning strategy, an issue referred to as the *determinacy* of the game in question.

An *integer game* of length ω is specified by the *payoff set* $A \subset \omega^\omega$. In the game, Players I and II alternate infinitely many times, each playing an integer in his turn. Player I wins if the infinite sequence they obtained belongs to the set A, otherwise Player II wins. Insignificant variations of this concept, which are nevertheless much more intuitive and easier to use, obtain when Players I and II can use moves from some other countable set in place of ω.

Fact 1.4.1. *[49] Games with Borel payoff set are determined. [20] If large cardinals exist then games with universally Baire payoff set are determined.*

A significant variation occurs if the players are allowed to choose their moves from a set larger than countable. Let U be an arbitrary set, and let $A \subset U^\omega$ be a set. The associated game with payoff A of length ω is played just as in the previous paragraph. To state the determinacy theorems, consider U^ω as a topological space with basic open neighborhoods of the form $O_t = \{\vec{u} \in U^\omega : t \subset \vec{u}\}$ as t varies over all finite sequences of elements of the set U.

Fact 1.4.2. *[48] Games with Borel payoff set are determined. Suppose that large cardinals exist, $A \subset U^\omega$ is a Borel set, $f : A \to X$ is a continuous function into a Polish space, and $B \subset X$ is a universally Baire set. The game with payoff set $f^{-1}B$ is determined, and moreover there is a winning strategy which remains winning in all set generic extensions.*

Still another significant variation occurs if the moves of the two players come from some fixed Polish space X and the game has α many rounds for some countable ordinal α. Consider the space X^α equipped with the standard Polish product topology.

Fact 1.4.3. *[55] (LC) Games with real entries, countable length, and universally Baire payoff set are determined.*

The games of longer than countable length are important and interesting, and in this book they appear in Section 6.1. However, I will never be concerned with their determinacy.

In numerous places I will refer to the Axiom of Determinacy (AD) and its variations, such as AD+, and the natural models for these axioms.

Definition 1.4.4. *The Axiom of Determinacy (AD) is the statement that integer games with arbitrary payoff set are determined. AD+ is the statement: every set of reals is ∞-Borel and games with ordinal entries, length ω, and payoff sets which are preimages of subsets of ω^ω under continuous maps $\mathrm{Ord}^\omega \to \omega^\omega$ are determined.*

Happily, I will never have to delve into the subtleties of AD+. Let me just state that it is an open question whether AD is in fact equivalent to AD+. In this book, I will need the following two pieces of information about the axiom AD+:

Fact 1.4.5. *Suppose that suitable large cardinals exist. Then $L(\mathbb{R}) \models$AD+. If Γ is a class of universally Baire sets closed under continuous preimages then $L(\mathbb{R})(\Gamma) \models$AD+.*

Fact 1.4.6. *[28] (ZF+DC+AD+) If $\kappa \in \Theta$ is a regular uncountable cardinal then there is a set $A \subset \omega^\omega$ and a prewellordering $\leq$ on A of length κ such that every analytic subset of A meets fewer than κ many classes.*

Here as usual Θ is the supremum of lengths of prewellorderings of the real numbers.

1.4.5 Forcing

The standard reference book for forcing terminology and basic facts is [29]. Suppose that $P, \leq$ is a partially ordered set, a *poset* for short. P is *separative* if for every $p, q \in P$, if every $r \leq p$ is compatible with q then $p \leq q$. The separative quotient of P is the partially ordered set of E-equivalence classes on P where pEq if every extension of p is compatible with q and vice versa, every extension of q is compatible with p, with the ordering inherited from the poset P. The separative quotient of P is separative. The posets considered in this book are generally not separative, and no effort is wasted on considering their separative quotients instead. Every separative poset P is isomorphic to a dense subset of a unique complete Boolean algebra denoted by $RO(P)$.

There is a historically and mathematically important forcing model mentioned in many places in the book, the *choiceless Solovay model*. Let me briefly outline its construction and basic features. Let κ be an inaccessible cardinal and $G \subset \text{Coll}(\omega, <\kappa)$ be a generic filter. Consider the submodel $M \subset V[G]$ consisting of those sets hereditarily definable in $V[G]$ from real parameters and parameters in the ground model. This is the definition of the choiceless Solovay model.

Fact 1.4.7. *The basic features of the Solovay model include*

1. *for every real number $r \in M$ the model M is a choiceless Solovay model over the model $V[r]$;*
2. *every set of reals is a wellordered union of length $\kappa = \omega_1^M$ of Borel sets.*

The book contains several isolated references to the *nonstationary tower forcing* Q_δ discovered by Woodin [79], recently exposed in [45]. If δ is a Woodin cardinal and $G \subset Q_\delta$ is a generic filter, then in $V[G]$ there is an elementary embedding $j : V \to M$ such that the model M is transitive, contains the same countable sequences of ordinals as $V[G]$, and $\omega_1^M = \delta$.

On several occasions I will refer to the Gandy–Harrington forcing [47]. This is the countable forcing of all nonempty lightface Σ^1_1 subsets of some fixed Polish space. As a countable forcing, this is similar to Cohen forcing; its worth derives