PART I

Basic Topics

1 Introduction

1.1 General

Trends have been seen in recent years in which engineers, physicists, and applied mathematicians perform research with interdisciplinary interactions. Vectors and tensors are the common language used in these interactions. Continuum mechanics combines mathematical operations and physical processes in continuous media for various subjects of engineering, physics, and applied mathematics. This is contrary to the notion that continuum mechanics is predominantly solid mechanics, as conceived in the past.

Thus our objective is to explore how the theory of continuum mechanics can combine deformations of solids, flows of inviscid and viscous fluids, electromagnetic waves, and motions of astrophysical objects in a single book. Common to all of these physical phenomena are the concepts of mass, velocity, acceleration, stress, momentum, and energy. We shall examine them as conceived in engineering disciplines and in spacetime of our universe, thus placing them all in proper perspective. Indeed, it shall be shown that the conservation forms of the electromagnetic continuum and Einstein's relativistic equations are similar to the conservation form of the Navier–Stokes system of equations such that the numerical solution schemes are similar for both cases, referring to a process of continuum mechanics dealing with speed of sound, speed of light, or both, in a similar fashion.

Newton's theory prevails today and will continue to govern our daily lives on earth. As the velocity of a particle increases, reaching nearly the speed of light, however, the effect of gravitation becomes significant and Newton's law must be replaced with Einstein's relativity theory. The word "continuum" implies "continuous media." Therefore, as long as mass, velocity, acceleration, stress, momentum, and energy are continuous functions in continuous media, our task of describing the behavior of these functions will be the same on earth and in the universe except as affected by relativity in spacetime and gravitation. Thus in this book our task is to become aware of both analogies and differences between them.

In continuum mechanics we are concerned with a macroscopic view, thus excluding the microscopic treatments such as in quantum mechanics and rarefied gases. A concept of fundamental importance here is that of the *mean free* *path*, which can be defined as the average distance a molecule travels between successive collisions with other molecules. The ratio of the mean free path λ to the characteristic length S of the physical boundaries of interest, called the *Knudsen number* Kn, may be used to determine the dividing line between the macroscopic and microscopic models:

$$\mathrm{Kn} = \frac{\lambda}{S} < 1, \quad \text{macroscopic}, \tag{1.1.1}$$

$$\mathrm{Kn} = \frac{\lambda}{S} \geq 1, \quad \text{microscopic}, \tag{1.1.2}$$

where $\lambda \cong 10^{-7}$ cm for solids and liquids and $\lambda \cong 10^{-6}$ cm for gases. Thus the Knudsen number is smaller than 1 for continuum mechanics as the characteristic length is larger than the mean free path. This is the case for solid mechanics, fluid dynamics, electromagnetic continuum, and relativistic continuum. It is known as the macroscopic problem as opposed to the microscopic problem such as in quantum mechanics (small characteristic length, $S \cong 10^{-33}$cm, large Knudsen number) or in rarefied gases (large mean free path, $\lambda > 10^{-6}$cm, large Knudsen number). Thus, in our study in continuum mechanics, rarefied gases and quantum mechanics are excluded.

For the macroscopic model, mass m is defined as a continuous function of volume Ω, such that density ρ is determined by the relation

$$\rho = \frac{dm}{d\Omega}, \tag{1.1.3}$$

whereas in the microscopic model, we define

$$\rho = \sum_{i=1}^{N} \rho_i = \sum_{i=1}^{N} m_i n_i, \tag{1.1.4}$$

where n_i denotes the number density of molecules per unit volume of a gas composed of a chemical species i.

The unified approach to the study of the global behavior of materials consists of, first, a thorough study of the basic principles common to all media and, second, a clear demonstration of the properties specific to the medium under consideration. The basic principles include the conservation of mass, the conservation of linear and angular momentum, the conservation of energy, and the principle of entropy. The underlying assumption of the unified theory is that these principles are valid for all materials irrespective of their constitution. Thus, to account for the nature of different materials – the various types of solids, liquids, or gases – we require additional equations, known as the equations of state, to describe the basic characteristics of the body and its response to the external agent under consideration.

This book is divided into two parts: Part I, Basic Topics, and Part II, Special Topics. We begin with the basic operations of vectors, matrices, Cartesian tensors, and domain and boundary surface integrals in Chap. 1, followed by kinematics, kinetics, linear elasticity, and Newtonian fluid mechanics in subsequent chapters of Part I. Curvilinear continuum, nonlinear continuum, electromagnetic continuum, and differential geometry continuum are included in Part II.

1.2 Vectors and Tensors in Cartesian Coordinates

A vector is determined in a given reference frame by a set of components. If a new coordinate system is introduced, the same vector is determined by another set of components, and these new components are related, in a definite way, to the old ones. The law of transformation of components of a vector is the essence of the vector representation.

Tensors are founded on a notion similar to that of vectors, but are much broader in conception. Tensor analysis is concerned with the study of *abstract objects*, called *tensors*, whose properties are independent of or invariant with the reference frames used to describe an object. A tensor is represented in a particular reference frame by a set of functions, termed its *components*, just as a vector is determined by a set of components. Tensor analysis deals with entities and properties that are independent of the choice of reference frames. Thus it forms an ideal tool for the study of natural laws because tensor equations are invariant with respect to a given category of coordinate transformations. Tensors are capable of delineating a variety of objects, ranging from scalars to multiple components of matter encountered in various physical phenomena. To discuss this subject further, however, it is necessary to introduce some notation and rules that will be applied to tensors and also to other topics in continuum mechanics.

Index Notation

A vector is denoted by a boldfaced letter symbol. A vector may be written in terms of its components by use of indices. For example, consider a vector in a right-handed rectangular three-dimensional Cartesian coordinate system (Fig. 1.2.1):

$$\mathbf{A} = A_1\mathbf{i}_1 + A_2\mathbf{i}_2 + A_3\mathbf{i}_3, \tag{1.2.1a}$$

where each $\mathbf{i}_i$ denotes one of the unit vectors and the indices $i = 1, 2, 3$ have a range of 3. This expression may be written as

$$\mathbf{A} = \sum_{i=1}^{3} A_i\mathbf{i}_i, \tag{1.2.1b}$$

where A_i indicates the components of the vector $\mathbf{A}$ at point p. Henceforth we shall dispense with the summation sign and write Eq. (1.2.1) in the form

$$\mathbf{A} = A_i\mathbf{i}_i \quad (i = 1, 2, 3)\,. \tag{1.2.1c}$$

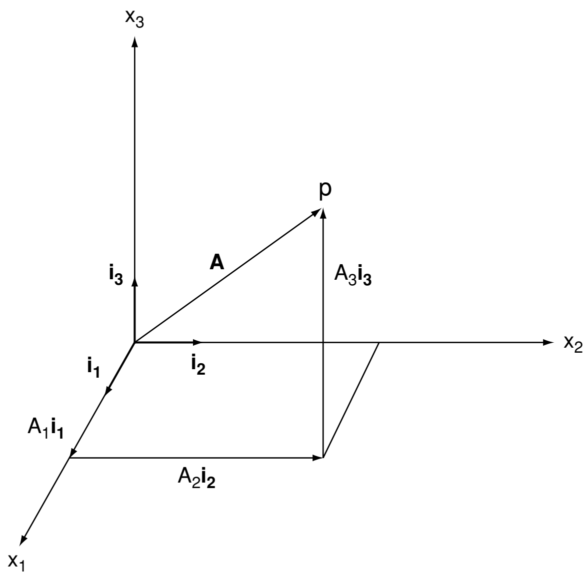

Figure 1.2.1. Three-dimensional representation of the vector **A** and its components.

Note that *repeated indices* (sometimes called *dummy indices*) imply summing over the range of the index. For example, let us consider

$$x_i' = a_{ij}x_j = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 \quad (i,\ j = 1, 2, 3)\,. \tag{1.2.2a}$$

Here, j is the repeated index and i must change independently of x_j to give x_1', x_2', x_3', which indicates that Eq. (1.2.2a) represents three equations. The index i, which is not repeated here, is called a *free index*, thus allowing the free index i in (1.2.2a) to assume $i = 1,\ 2,\ 3$ one at a time and to obtain a total of three equations:

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \\ x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3, \\ x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3. \end{aligned} \tag{1.2.2b}$$

Vector Multiplication

The Kronecker delta is defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}. \tag{1.2.3a}$$

This represents the 3×3 unit matrix,

$$\delta_{ij} = \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [\mathbf{I}], \tag{1.2.3b}$$

where [**I**] is the unity matrix.

The permutation symbol is given by

$$\epsilon_{ijk} = \begin{cases} 1 & \text{for even permutations of } ijk \text{ (123, 231, 312),} \quad \text{clockwise rotation} \\ -1 & \text{for odd permutations of } ijk \text{ (132, 213, 321),} \quad \text{counterclockwise rotation.} \\ 0 & \text{for two or more equal indices (112, 111, etc.)} \end{cases} \tag{1.2.4}$$

Note that the permutation symbol has an array (9×3) of 27 terms, with only 6 terms being nonzero and 21 terms zero.

A dot product of any two vectors in index notation reads

$$\mathbf{A} \cdot \mathbf{B} = A_i \mathbf{i}_i \cdot B_j \mathbf{i}_j = A_i B_j \delta_{ij} = A_i B_i = A_j B_j = \lambda. \tag{1.2.5}$$

Because of the orthogonal or orthonormal coordinate system, a dot product of the unit vectors produces a Kronecker delta:

$$\mathbf{i}_i \cdot \mathbf{i}_j = \delta_{ij}. \tag{1.2.6}$$

The role of the Kronecker delta is to interchange the index of a component of a vector, as demonstrated in Eq. (1.2.5). In this process, as a consequence of the Kronecker delta, only the nonzero terms are allowed to remain, the zero terms being removed. Furthermore, the dot product of two vectors results in a scalar λ, which does not have a free index.

On the other hand, a cross product of any two vectors in index notation reads

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= A_i \mathbf{i}_i \times B_j \mathbf{i}_j \\ &= A_i B_j \mathbf{i}_i \times \mathbf{i}_j = A_1 B_1 \mathbf{i}_1 \times \mathbf{i}_1 + A_1 B_2 \mathbf{i}_1 \times \mathbf{i}_2 + A_2 B_1 \mathbf{i}_2 \times \mathbf{i}_1 + \cdots + \\ &= 0 + A_1 B_2 \mathbf{i}_3 - A_2 B_1 \mathbf{i}_3 \\ &\quad + \cdots + (27 \text{ terms, } \ 21 \text{ of them zero, } \ \text{only 6 terms nonzero}) \\ &= A_i B_j \epsilon_{ijk} \mathbf{i}_k \\ &= (A_2 B_3 - A_3 B_2)\mathbf{i}_1 + (A_3 B_1 - A_1 B_3)\mathbf{i}_2 + (A_1 B_2 - A_2 B_1)\mathbf{i}_3, \end{aligned} \tag{1.2.7}$$

where the cross product of unit vectors produces a permutation symbol ϵ_{ijk} such that

$$\mathbf{i}_i \times \mathbf{i}_j = \epsilon_{ijk} \mathbf{i}_k. \tag{1.2.8}$$

It is interesting to note that the result obtained in Eq. (1.2.7) can be written as

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= \begin{vmatrix} \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \\ &= (A_2 B_3 - A_3 B_2)\mathbf{i}_1 - (A_1 B_3 - A_3 B_1)\mathbf{i}_2 + (A_1 B_2 - A_2 B_1)\mathbf{i}_3, \end{aligned} \tag{1.2.9}$$

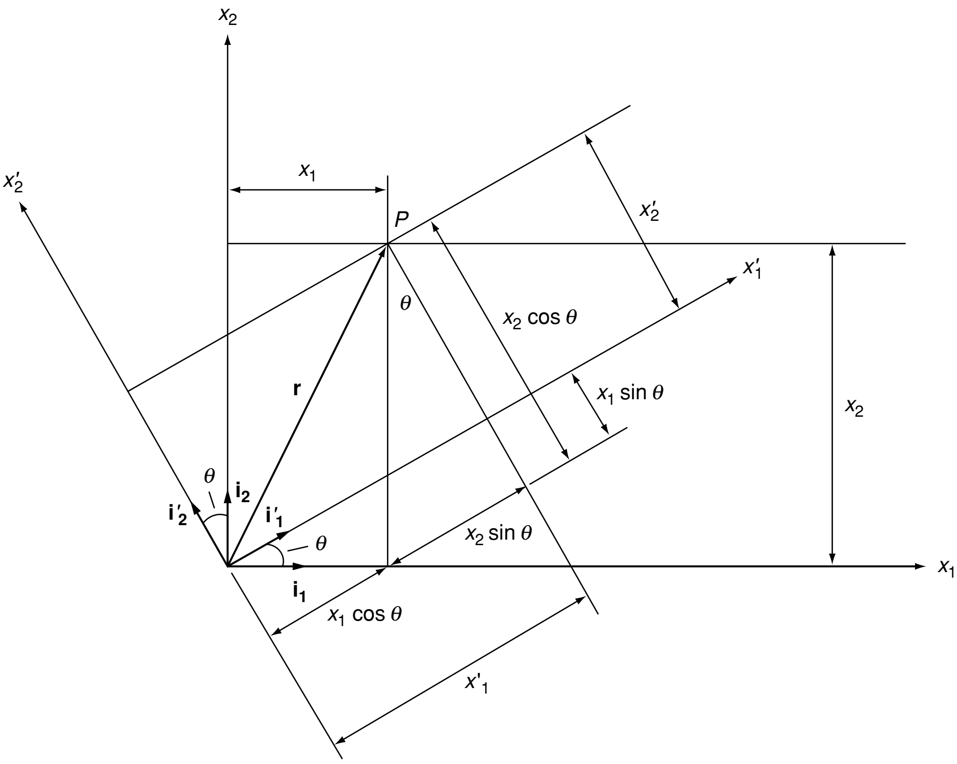

Figure 1.2.2. Coordinate transformation rotation between x_i and x_i' coordinates by an angle θ.

indicating that the application of a permutation symbol results in the determinant of a 3×3 matrix array. Notice that Eq. (1.2.7) indicates a logical sequence of derivation whereas Eq. (1.2.9) is merely a definition resulting from Eq. (1.2.7).

It can easily be shown that $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$ (commutative law) and $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$. The reader is encouraged to prove these relations by using the index notation as an exercise.

Tensors

The quantities that appear in the foregoing paragraphs are identified as tensors because they satisfy the basic properties set forth at the beginning of this section. To provide a specific example, let us examine Eqs. (1.2.2b). Let $\mathbf{r}$ be a position vector in Fig. 1.2.2:

$$\mathbf{r} = x_i \mathbf{i}_i \quad (i = 1, 2, 3), \tag{1.2.10a}$$

where the standard right-hand rule is used. If the old coordinates x_i are rotated by an angle θ about the x_3 axis to a set of new coordinates x_i', then the position vector can be represented by

$$\mathbf{r} = x_i' \mathbf{i}'_i \quad (i = 1, 2, 3). \tag{1.2.10b}$$

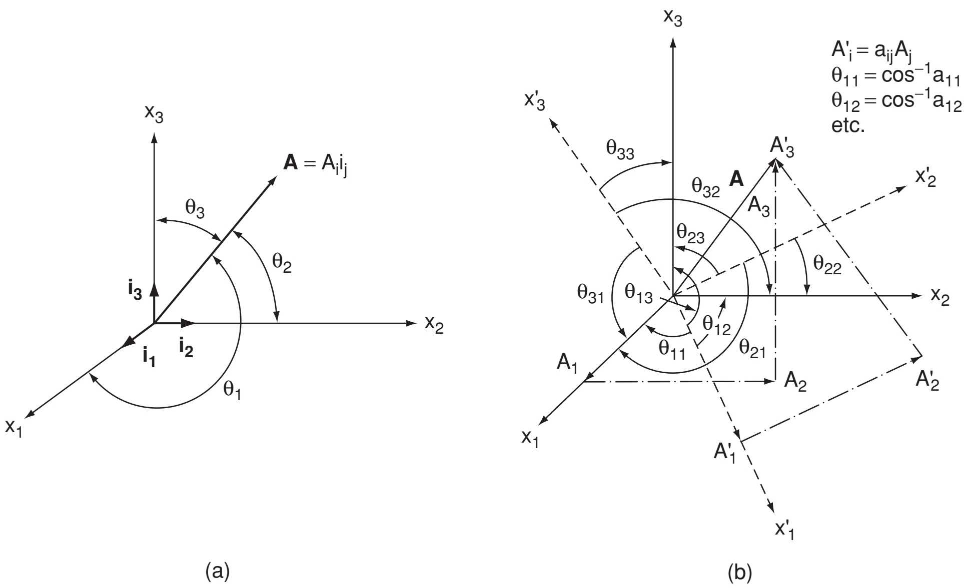

Figure 1.2.3. Coordinate transformations: (a) representation of a vector **A**, (b) representation of a vector **A** in two sets of right-handed Cartesian axes with different orientations.

Figure 1.2.2 shows that the old coordinates x_i are rotated about the x_3 axis counterclockwise by an angle θ to a set of new coordinates x'_i. We can calculate the new coordinates in terms of the old coordinates by adding and subtracting coordinate lengths,

$$\begin{aligned} x'_1 &= (\cos\theta)x_1 + (\sin\theta)x_2, \\ x'_2 &= -(\sin\theta)x_1 + (\cos\theta)x_2, \\ x'_3 &= x_3. \end{aligned} \tag{1.2.11}$$

These relationships may be obtained in a more general and systematic manner. To this end, we write Eqs. (1.2.11) in a form given by Eq. (1.2.2a) and as shown in Fig. 1.2.3:

$$x'_i = a_{ij}x_j \quad (i, j = 1, 2, 3), \tag{1.2.12}$$

with

$$\begin{aligned} a_{11} &= \cos\theta_{11} = \cos\theta, \\ a_{12} &= \cos\theta_{12} = \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta, \\ a_{13} &= \cos\frac{\pi}{2} = 0, \\ a_{21} &= \cos\theta_{21} = \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta, \end{aligned}$$

$$\begin{aligned}
a_{22} &= \cos\theta_{22} = \cos\theta, \\
a_{23} &= \cos\frac{\pi}{2} = 0, \\
a_{31} &= \cos\theta_{31} = \cos\frac{\pi}{2} = 0, \\
a_{32} &= \cos\theta_{32} = \cos\frac{\pi}{2} = 0, \\
a_{33} &= \cos\theta_{33} = \cos 0 = 1,
\end{aligned} \tag{1.2.13}$$

where the subscripts on the rotation angle θ_{ab} imply that an angle is measured from the new axis a (x_i' axis) to the old axis b (x_j axis).

In matrix notation, using the results of Eqs. (1.2.13), we write Eq. (1.2.12) as

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \tag{1.2.14}$$

This is the same as in Eqs. (1.2.11), which were obtained from a geometrical deduction. The approach given in Eqs. (1.2.13) is much more efficient, particularly when rotations about various axes are performed many times consecutively, as shown in Example 1.2.1. Note that the old coordinates x_j are transformed into the new coordinates x_i' by means of the quantities called the *transformation matrix* a_{ij}, whose components are the cosines of angles measured from the new coordinates x_i' to the old coordinates x_j. We have seen that the position vector $\mathbf{r}$ remains invariant through coordinate transformations. The quantities x_i', x_j, and a_{ij} are the abstract objects whose properties remain invariant with the coordinate transformations. Therefore they are all tensors, as defined earlier.

Similarly, we may consider a vector $\mathbf{A} = A_i\mathbf{i}_i$ oriented at θ_1, θ_2, and θ_3 from the respective rectangular Cartesian coordinate axes [Fig. 1.2.3(a)] so that

$$A_i = An_i \quad (i = 1, 2, 3),$$

where

$$A = \sqrt{A_1^2 + A_2^2 + A_3^2},$$

$$n_1 = \frac{A_1}{A} \quad n_2 = \frac{A_2}{A} \quad n_3 = \frac{A_3}{A}.$$

Here $n_i(n_1, n_2, n_3)$ are the direction cosines whose properties must satisfy

$$\mathbf{n}\cdot\mathbf{n} = n_in_i = n_1^2 + n_2^2 + n_3^2 = 1.$$

Note also in Eq. (1.2.12),

$$\begin{aligned}
\mathbf{n}^{(1)}\cdot\mathbf{n}^{(1)} &= a_{11}^2 + a_{12}^2 + a_{13}^2 = \cos^2\theta + \sin^2\theta = 1, \\
\mathbf{n}^{(2)}\cdot\mathbf{n}^{(2)} &= a_{21}^2 + a_{22}^2 + a_{23}^2 = (-\sin\theta)^2 + \cos^2\theta = 1, \\
\mathbf{n}^{(3)}\cdot\mathbf{n}^{(3)} &= a_{31}^2 + a_{32}^2 + a_{33}^2 = 1.
\end{aligned}$$

Proceeding in a similar manner, as shown in Fig. 1.2.3(b), we may write

$$A'_i = a_{ij} A_j,$$

where A'_i refers to a set of new coordinates so that the vector **A** is now measured in terms of the new coordinates and a_{ij} acts as a set of direction cosines, with i and j indicating the new and old coordinates, respectively.

Again, the vector components A'_i, A_j as well as a_{ij}, are tensors. They are abstract objects invariant with the frame of reference. Once a quantity is determined to be a tensor, then the number of free indices indicates the order of the tensor. Thus we define, in general, for any abstract object B,

$$\begin{aligned} B &= \text{zero-order tensor},\\ B_i &= \text{first-order tensor},\\ B_{ij} &= \text{second-order tensor},\\ B_{ijk} &= \text{third-order tensor},\\ B_{ijkl} &= \text{fourth order tensor},\\ &\vdots\\ \text{etc.} \end{aligned}$$

Tensors using the Cartesian coordinates are called "Cartesian tensors," whereas those based on the curvilinear coordinates are called "curvilinear tensors," which will be introduced in Chap. 6. Note that the second-order tensor refers to the 3×3 matrix as identified such as in Eqs. (1.2.3b) and (1.2.14).

EXAMPLE 1.2.1. Consider a set of new axes x'_i as obtained by rotating the old axes x_i through a 60° angle counterclockwise about the x_2 axis. What are the components of a vector **A** in the new coordinates if A_i in the old coordinates are (2, 1, 3)?

Solution. The direction cosines a_{ij} between the old and new axes are

$$a_{ij} = \begin{bmatrix} \cos 60^\circ & 0 & -\sin 60^\circ \\ 0 & 1 & 0 \\ \sin 60^\circ & 0 & \cos 60^\circ \end{bmatrix}.$$

From the transformation law,

$$A'_i = a_{ij} A_j,$$

we obtain

$$A'_1 = a_{11}A_1 + a_{12}A_2 + a_{13}A_3 = \frac{2 - 3\sqrt{3}}{2},$$

$$A'_2 = a_{21}A_1 + a_{22}A_2 + a_{23}A_3 = 1,$$

$$A'_3 = a_{31}A_1 + a_{32}A_2 + a_{33}A_3 = \frac{2\sqrt{3} + 3}{2},$$