Purity, Spectra and Localisation

This book is an account of a fruitful interaction between algebra, mathematical logic, and category theory. It is possible to associate a topological space to the category of modules over any ring. This space, the Ziegler spectrum, is based on the indecomposable pure-injective modules. Although the Ziegler spectrum arose within the model theory of modules and plays a central role in that subject, this book concentrates on its algebraic aspects and uses.

The central aim is to understand modules and the categories they form through associated structures and dimensions which reflect the complexity of these, and similar, categories. The structures and dimensions considered arise through the application of ideas and methods from model theory and functor category theory. Purity and associated notions are central, localisation is an ever-present theme and various types of spectrum play organising roles.

This book presents a unified account of material which is often presented from very different viewpoints and it clarifies the relationships between these various approaches. It may be used as an introductory graduate-level text, since it provides relevant background material and a wealth of illustrative examples. An extensive index and thorough referencing also make this book an ideal, comprehensive reference.

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To Abigail, Stephen and Iain

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Mike Prest
Frontmatter
More Information

Contents

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Mike Prest
Frontmatter
More Information

Preface

In his paper [726], on the model theory of modules, Ziegler associated a topological space to the category of modules over any ring. The points of this space are certain indecomposable modules and the definition of the topology was in terms of concepts from model theory. This space, now called the Ziegler spectrum, has played a central role in the model theory of modules. More than one might have expected, this space and the ideas surrounding it have turned out to be interesting and useful for purely algebraic reasons. This book is mostly about these algebraic aspects.

The central aim is a better understanding of the category of modules over a ring. Over most rings this category is far too complicated to describe completely so one must be content with aiming to classify the most significant types of modules and to understand more global aspects in just a broad sense, for example by finding some geometric or topological structure that organises some aspect of the category and which reflects the complexity of the category.

By "significant types of modules" one might mean the irreducible representations or the "finite" (finite-dimensional/finitely generated) ones. Here I mean the pure-injective modules. Over many rings this class of modules includes, directly or by proxy, the "finite" ones. There is a decomposition theorem which means that for most purposes we can concentrate on the indecomposable pure-injective modules.

The Ziegler spectrum is one example of an "organising" structure; it is a topological space whose points are the isomorphism classes of indecomposable pure-injectives, and the Cantor–Bendixson analysis of this space does reflect various aspects of complexity of the module category. There are associated structures: the category of functors on finitely presented modules; the lattice of pp conditions; the presheaf of rings of definable scalars. Various dimensions and ranks are defined on these and they are all linked together.

Here I present a cluster of concepts, techniques, results and applications. The inputs are from algebra, model theory and category theory and many of the results and methods are hybrids of these. The way in which these combine here is something which certainly I have found fascinating. The applications are mainly algebraic though not confined to modules since everything works in good enough abelian categories. Again, I have been pleasantly surprised by the extent to which what began in model theory has had applications and ramifications well beyond that subject.

Around 2000 it seemed to me that the central part of the subject had pretty well taken shape, though mainly in the minds of those who were working with it and using it. Much was not written down and there was no unified account so, foolhardily, I decided to write one. This book, which is the result, has far outgrown my original intentions (in length, time, effort, . . .). In the category of books it is a pushout of a graduate-level course and a work of reference.

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Mike Prest
Frontmatter
More Information

Introduction

The Ziegler spectrum, Zg_R, of a ring R is a topological space. It is defined in terms of the category of R-modules and, although a Ziegler spectrum can be assigned to much more general categories, let us stay with rings and modules at the beginning. The points of Zg_R are certain modules, more precisely they are the isomorphism types of indecomposable pure-injective (also called algebraically compact) right R-modules. Any injective module is pure-injective but usually there are more, indeed a ring is von Neumann regular exactly if there are no other pure-injective modules (2.3.22). If R is an algebra over a field k, then any module which is finite-dimensional as a k-vector space is pure-injective (4.2.6). Every finite module is pure-injective (4.2.6). Another example is the ring of p-adic integers, regarded as a module over any ring between $\mathbb{Z}$ and itself (4.2.8). The pure-injective modules mentioned so far are either "small" or, although large in some sense, have some kind of completeness property. There is something of a general point there but, as it stands, it is too vague: not all "small" modules are pure-injective. For example, the finite-length modules over the first Weyl algebra, $A_1(k)$, over a field k of characteristic zero are not pure-injective (8.2.35). They are small in the sense of being of finite length, but large in that they are infinite-dimensional. Nevertheless each indecomposable finite-length module over the first Weyl algebra has a pure-injective hull (a minimal pure, pure-injective extension, see Section 4.3.3) which is indecomposable. Indeed, associating to a finite-length module its pure-injective hull gives a bijection between the set of (isomorphism types of) indecomposable finite-length modules over $A_1(k)$ and a subset of the Ziegler spectrum of $A_1(k)$ (8.2.39).

Ziegler defined the topology of this space in terms of solution sets to certain types of linear conditions (5.1.21) but there are equivalent definitions: in terms of morphisms between finitely presented modules (5.1.25); also in terms of finitely presented functors (10.2.45). Ziegler showed that understanding this space, in the very best case obtaining a list of points and an explicit description of the topology, is the key to answering most questions about the model theory of modules over the given ring. For that aspect one may consult [495]. Most subsequent advances have been driven more by algebraic than model-theoretic questions though much of what is here can be reformulated to say something about the model theory of modules.

Over some rings there is a complete description of the Ziegler spectrum (see especially Section 5.2 and Chapter 8); for $R = \mathbb{Z}$ the list of points is due to Kaplansky [330], see Section 5.2.1.

A module which is of finite length over its endomorphism ring is pure-injective (4.2.6, 4.4.24) but, unless the ring is right pure-semisimple (conjecturally equivalent to being of finite representation type, see Section 4.5.4), one should expect there to be "large" points of the Ziegler spectrum. Over an artin algebra a precise expression of this is the existence of infinite-dimensional indecomposable pure-injectives (5.3.40) if the ring is not of finite representation type. This is an easy consequence of compactness of the Ziegler spectrum of a ring (5.1.23). Even if one is initially interested in "small" modules, for example, finite-dimensional representations, the "large" modules may appear quite naturally: often the latter parametrise, in some sense or another, natural families of the former (e.g. 5.2.2, Sections 4.5.5 and 15.1.3). Examples of such large parametrising modules are the generic modules of Crawley-Boevey (Section 4.5.5).

A natural context for most of the results here is that of certain, "definable", subcategories of locally finitely presented abelian categories: the latter are, roughly, abelian categories in which objects are determined by their "elements", see Chapter 16. There are reasons for working in the more general context beyond simply wider applicability. Auslander and coworkers, in particular Reiten, showed how, if one is interested in finite-dimensional representations of a finite-dimensional k-algebra, it is extremely useful to move to the, admittedly more abstract, category of k-linear functors from the category of these representations to the category of k-vector spaces. It turns out that, in describing ideas around the Ziegler spectrum, moving to an associated such functor category often clarifies concepts and simplifies arguments (and leads to new results!). By this route one may also dispense with the terminology of model theory, though model theory still provides concepts and techniques, and replace it with the more widely known terminology of categories and functors. In particular, one may define the Ziegler spectrum of a ring as a topology on the (set of isomorphism types of) indecomposable injectives in the corresponding functor category (12.1.17). This topology is dual (Section 5.6) to another topology, on the same set, which one might regard as the (Gabriel–)Zariski spectrum of the functor category (14.1.6).

This equivalence between model-theoretic and functorial methods is best explained by an equivalence, 10.2.30, between the model-theoretic category of "imaginaries" (in the sense of Section B.2) and the category of finitely presented functors. There is some discussion of this equivalence between methods, and see Appendix C, but I have tended to avoid the terminology of model theory, except where I consider it to be particularly efficient or where there is no algebraic equivalent. That is simply because it is less well known, though I hope that one effect of this book will be that some people become a little more comfortable with it. This does mean that those already familiar with model theory might have to work a bit harder than they expected: the terminological adjustments are quite slight, but the conceptual adjustments (the use of functorial methods) may well require more effort. It should be noted that much of the relevant literature does assume familiarity with the most basic ideas from model theory.

As mentioned already, it is possible to give definitions (5.1.1, 5.1.25) of the Ziegler spectrum of a ring purely in terms of its category of modules, that is, without reference to model theory or to any "external" (functor) category. In fact the book begins by taking a "naïve", element-wise, view of modules and gradually, though not monotonically, takes an increasingly "sophisticated" view. I discuss this now.

I believe that there is some advantage in beginning in the (relatively) concrete context of modules and, consequently, in the first part of the book, modules are simply sets with structure and most of the action takes place in the category of modules. Many results are presented, or at least surveyed, in that first part, so a reader may refer to these without having to absorb the possibly unfamiliar functorial point of view.

Nevertheless, it was convincingly demonstrated by Herzog in the early 1990s that the most efficient and natural way to prove Ziegler's results, and many subsequent ones, is to move to the appropriate functor category. Indeed it was already appreciated that work, particularly of Gruson and Jensen, ran, in places, parallel to pre-Ziegler results in the model theory of modules and some of the translation between the two languages (model-theoretic and functorial) was already known.

Furthermore, many applications have been to the representation theory of finite-dimensional algebras, where functorial methods have become quite pervasive. So, at the beginning of Part II, we move to the functor category. In fact, the ground will have been prepared already, in the sense that I call on results from Part II in more than a few proofs in Part I. The main reason for this anticipation, and consequent complication in the structure of logical dependencies in the book, is that I wish to present the functorial proofs of many of the basic results. The original model-theoretic and/or algebraic ("non-functorial") proofs are available elsewhere, whereas the functorial proofs are scattered in the literature and, in many cases, have not appeared.

In the second part of the book modules become certain types of functors on module categories: in the third part they become functors on functor categories. This third part deals with results and questions clustering around relationships between definable categories. Model theory reappears more explicitly in this part because it is, from one point of view, all about interpretability of one category in another.

One could say that Part I is set mostly in the category Mod-R, that Part II is set in the functor category $(R\text{-mod}, \mathbf{Ab})$ and that Part III is set in the category $\big((R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}, \mathbf{Ab}\big)$. The category, $\mathrm{Ex}\big((R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}, \mathbf{Ab}\big)$, of exact functors on the category $(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ appears and reappears in many forms throughout the book.

A second spectrum also appears. The Zariski spectrum is well known in the context of commutative rings: it is the space of prime ideals endowed with the Zariski topology. It is also possible to define this space, *à la* Gabriel [202], in terms of the category of modules, namely as the set of isomorphism types of indecomposable injectives endowed with a topology which can be defined in terms of morphisms from finitely presented modules. That definition makes sense in the category of modules over any ring, indeed in any locally finitely presented abelian category. Applied to the already-mentioned functor category $(R\text{-mod}, \mathbf{Ab})$, one obtains what I call the Gabriel–Zariski spectrum. It turns out that this space can also be obtained from the Ziegler spectrum, as the "dual" topology which has, for a basis of open sets, the complements of compact open sets of the Ziegler topology.

Much less has been done with this than with the Ziegler topology and I give it a corresponding amount of space. I do, however, suspect that there is much to discover about it and to do with it. The Gabriel–Zariski spectrum has a much more geometric character than the Ziegler spectrum, in particular it carries a sheaf of rings which generalises the classical structure sheaf: it is yet another "non-commutative geometry".

Chapters 1–5 and 10–12, minus a few sections, form the core exposition. The results in the first group of chapters are set in the category of modules and lead the reader through pp conditions and purity to the definition and properties of the Ziegler spectrum. The methods used in the proofs change gradually; from elementary linear algebra to making use of functor categories. One of my reasons for writing this book, rather than being content with what was already in the literature, was to present the basic theory using these functorial methods since they have lead to proofs which are often much shorter and more natural than the original ones. The second group of chapters introduces those methods, so the reader of Chapters 1–5 must increasingly become the reader of Chapters 10–12.

Beyond this core, further general topics are presented in Chapters 6 and 14 (rings and sheaves of definable scalars, the Gabriel–Zariski topology) and in

Chapters 7 and 13 (dimensions). Chapter 9, on ideals in mod-R, leads naturally to the view of Part II.

Already that gives us a book of some 500 pages, yet there is much more which should be said, not least applications in specific contexts. At least some of that is said in the remaining pages, though often rather briefly. Chapter 8 contains examples and descriptions of Ziegler spectra over various types of ring. Some of the most fruitful development has been in the representation theory of artin algebras (Chapter 15). The theory applies in categories much more general than categories of modules and Chapters 16 and 17 present examples. In these chapters the emphasis is more on setting out the basic ideas and reporting on what has been done, so rather few proofs are given and the reader is referred to the original sources for the full story.

Though the book begins with systems of linear equations, by the time we arrive at Part III we are entering very abstract territory, an additive universe which parallels that of topos theory. Chapter 18 introduces this, though not at great length since this is work in progress and likely not to be in optimal form.

Ziegler's paper was on the model theory of modules and, amongst all this algebraic development, we should not forget the open questions and developments in that subject, so Appendix D is a, very brief, update on the model theory of modules per se.

Beyond this, there is background on model theory in Appendices A and B, as well as general background (Appendix E) and a model theory/functor category theory "dictionary" (Appendix C).

Relationship with the earlier book and other work As to the relationship of this book with my earlier one *Model Theory and Modules* [495], this is, in a sense, a sequel but the emphases of the two books are very different. The earlier book covered model-theoretic aspects of modules and related algebraic topics, and it was written from a primarily model-theoretic standpoint. In this book the viewpoint is algebraic and category-theoretic though it is informed by ideas from model theory. No doubt the model theory proved to be an obstacle for some readers of [495] and perhaps the functor-category theory will play a similar role in this book. But I hope that by introducing the functorial ideas gradually through the first part of this book I will have made the path somewhat easier. Readers who have some familiarity with the contents of [495] will find here new results and fresh directions and they will find that the text reflects a great change in viewpoint and expansion of methods that has taken place in the meantime.

The actual overlap between the books is rather smaller than one might expect, given that they are devoted to the same circle of ideas. Part of the reason for this is that many new ideas and results have been produced in the intervening years.

If that were all, then an update would have sufficed. But one of the reasons for my writing this book is to reflect the fundamental shift in viewpoint, the adoption of a functor-category approach, that has taken place in the area. Although, in this context, the languages of model theory and functor-category theory are, in essence and also in many details, equivalent (indeed, I provide, at Appendix C, a dictionary between them) there have proved to be many conceptual advantages in adopting the latter language. Readers familiar with the arguments of [495] or Ziegler's paper, will see here how complex and sometimes apparently ad-hoc arguments become natural and easy in this alternative language. That is not to say that the insights and techniques of model theory have been abandoned. In fact they inform the whole book, although this might not always be apparent. Some model-theoretic ideas have been explicitly retained, for example the notion of pp-type, because we need this concept and because there is no algebraic name for it. On the other hand, there is no need in this book to treat formulas as objects of a formal language, so I refer to them simply as conditions (which is, in any case, how we think of them).

In the more model-theoretic approach there was a conscious adaptation of ideas from module theory (at least on my part, see, for example, [495, p. 173], and, I would guess, also by Garavaglia, see especially [209]), using the heuristic that pp conditions are generalised ring elements, that pp-types are generalised ideals (in their role as annihilators) and that various arguments involving positive quantifier-free types in injective modules extend to pp-types in pure-injective modules. Moving to the functor category has the effect of turning this analogy into literal generalisation. See, for example, the two proofs of 5.1.3 bearing in mind the heuristic that a pp-type is a generalised right ideal.

Various papers and books contain significant exposition of some of the material included here. Apart from my earlier book, [495], and some papers, [493], [497], [503], [511], there are Rothmaler's, [620], [622], [623], the survey articles [61], [484], [568], [514], [728], a large part of the book, [323], of Jensen and Lenzing, the monograph, [358], of Krause and sections in the books of Facchini, [183], and Puninski, [564]. There is also the more recent [516] which deals with the model theory of definable additive categories.

In this book I have tried to include at least mention of all recent significant developments but, in contrast to the writing of [495], I have tried, with a degree of success, to restrain my tendency to aim to be encylopaedic. There are some topics which I just mention here because, although I would have liked to have said more, I do not have the expertise to say anything more useful that what can easily be found already in the literature. These include: the work of Guil Asensio and Herzog developing a theory of purity in Flat-R (see the end of Section 4.6); recent and continuing developments around cotilting modules and cotorsion theories (see the end of Section 18.2.3); work of Beligiannis and others around triangulated categories and generalisations of these (see Chapter 17).

Thanks, especially, to: Ivo Herzog who, in the early 1990s, showed me how Ziegler's arguments become so much easier and more natural in the functor category; Kevin Burke, Bill Crawley-Boevey and Henning Krause who convinced me in various ways of the naturality and power of the functor category idiom and of the desirability of adopting it.

I am also grateful to a number of colleagues, in particular to Gena Puninski, for comments on a draft of this.

General comments Occasionally I define a concept or prove a result that is defined or proved elsewhere in the book. Oversights excepted, this is to increase the book's usefulness as a reference. For example, if a concept is defined using the functor-category language, or in a very general context, it can be useful to have an alternative definition, possibly in a more particular context, also to hand.

I also play the following mean trick. In Part I, R is a ring and modules are what you think they are. In Part II, I reveal that R was, all the time, a small preadditive category and that what you thought were modules were actually functors. To have been honest right through Part I would have made the book even harder to read. In any case, you have now been warned.

Bibliography As well as containing items which are directly referenced, the bibliography is an "update" on that in [495], so contains a good number of items which do not occur in the bibliography of [495] but which continue or are relevant to some of the themes there. In particular I have tried to be comprehensive as regards including papers which fall within the model theory of modules, even though I have only pointed to the developments there.

Conventions and notations

Conventions A module will be a right module if it matters and if the contrary is not stated.

Tensor product $\otimes$ means, unless indicated otherwise, tensor product over R, $\otimes_R$, where R is the ring of the moment.

By a functor between preadditive categories is meant an additive functor even though "additive" is hardly ever said explicitly.

"Non-commutative" means "not necessarily commutative".

To say that a tuple of elements is **from** M is to say that each entry of the tuple is an element of M (the less accurate "in" in place of "from" may have slipped through in places).

The value of a function f (or functor F) on an element a (or object A or morphism g) is usually denoted fa (respectively FA, Fg) but sometimes, for clarity or to avoid ambiguity, $f(a)$ or $f \cdot a$ (and similarly).

I make no distinction between monomorphisms and embeddings and my use of the terms is determined by a mixture of context and whim: similarly for map, morphism and homomorphism.

In the context of categories I write $\mathcal{A} \simeq \mathcal{B}$ if the categories $\mathcal{A}$ and $\mathcal{B}$ are naturally equivalent. Sometimes the stronger relation of isomorphism holds but we don't need a different symbol for that.

Equality, "=", often is used where $\simeq$ (isomorphism or natural equivalence) would be more correct, especially, but not only, if our choice of copy has not yet been constrained.

I write $l(\overline{a})$ for the **length** (number of entries) of a tuple $\overline{a}$. By default a tuple has length "n" and this explains (I hope all) unheralded appearances of this symbol. If $\overline{a}$ is a tuple, then its typical coordinate/entry is a_i.

Tuples and matrices are intended to match when they need to. So, if $\overline{a}, \overline{b}$ are tuples, then the appearance of "$\overline{a} + \overline{b}$" implies that they are assumed to have the same length, and the expression means the tuple with ith coordinate equal to $a_i + b_i$. Similarly, if $\overline{a}$ and $\overline{b}$ are tuples and H is a matrix, then writing "$(\overline{a}\,\overline{b})H = 0$" implies that the number of rows of H is the length of $\overline{a}$ plus the length of $\overline{b}$. The notations $(\overline{a}\,\overline{b})$, $(\overline{a}, \overline{b})$ and $\overline{a}\overline{b}$ are all used for the tuple whose entries are the entries of $\overline{a}$ followed by those of $\overline{b}$.

If X is a subset of the R-module M^n and W is a subset of R^n, then by $X \cdot W$ we mean the subgroup of M generated by the $(a_1, \ldots, a_n) \cdot (r_1, \ldots, r_n) = \sum_{i=1}^n a_i r_i$ with $(a_1, \ldots, a_n) \in X$ and $(r_1, \ldots, r_n) \in W$. If, on the other hand, $X \subseteq M$ and $W \subseteq R^n$, then $X \cdot W$ will mean the subgroup of M^n generated by the $(ar_1, \ldots, ar_n)$ with $a \in M$ and $(r_1, \ldots, r_n) \in W$. Sometimes the "·" will be omitted. So read this as "dot product" if that makes sense, otherwise diagonal product.

For matrices I use notation such as $(r_{ij})_{ij}$ meaning that i indexes the rows, j the columns and the entry in the ith row and jth column is r_{ij}. I also use, I hope self-explanatory, partitioned-matrix notation in various places.

Notation such as $(M_i)_i$ is used for indexed sets of objects and I tend to refer to them (loosely and incorrectly – there may be repetitions) as sets.

If I say that a ring is noetherian I mean right and left noetherian: similarly with other naturally one-sided conditions.

The "radical" of a ring R means the Jacobson radical and this is denoted $J(R)$ or just J. The notation "rad" is used for the more extended (to modules, functors, categories) notion.

If $\mathcal{C}$ is a category, then a class of objects of $\mathcal{C}$ is often identified with the full subcategory which has these as objects.

The notation (co)ker(f) and term (co)kernel are used for both the object and the morphism.

The notation $^\perp$ is used for both Hom- and Ext-orthogonality, depending on context.

The ordering on pp-types is by solution sets, so I write $p \geq q$ if $p(M) \geq q(M)$ for every M. Therefore $p \geq q$ iff $p \subseteq q$ (the latter inclusion as sets of pp conditions). This is the opposite convention from that adopted in [495]. There are arguments for both but I think that the convention here makes more sense.

If χ is a condition and $\overline{v}$ is a sequence of variables, then writing $\chi(\overline{v})$ implies that the free variables of χ all occur in $\overline{v}$ but not every component variable of $\overline{v}$ need actually occur in χ.

I will be somewhat loose regarding the distinction between small and skeletally small categories, for instance stating a result for small categories but applying it to skeletally small categories such as mod-R.

The term "tame (representation type)" sometimes includes finite representation type, but not always; the meaning should be clear from the context.

The term "torsion theory" will mean hereditary torsion theory unless explicitly stated otherwise.

Morphisms in categories compose to the left, so fg means do g then f. So, in the category $\mathbb{L}_R^{\mathrm{eq}+}$ associated to the category of right R-modules, pp-defined maps compose to the left. The action of these maps on right R-modules is, however, naturally written on the right, so the ring of definable scalars of a right R-module is the *opposite* of a certain endomorphism ring in the corresponding localisation of this category. I have tried to be consistent in this but there may be places where an $^{\mathrm{op}}$ should be inserted.

The term "preprint" is used to mean papers which are in the public domain but which might or might not be published in the future.

Notations The notation $\mathbb{Z}_n$ is used for the factor group $\mathbb{Z}/n\mathbb{Z}$. The localisation of $\mathbb{Z}$ at p is denoted by $\mathbb{Z}_{(p)}$ and the p-adic integers by $\overline{\mathbb{Z}_{(p)}}$.

End(A) denotes the endomorphism ring of A and Aut(A) its automorphism group. The group, Hom(A, B), of morphisms from A to B is usually abbreviated to (A, B). Similar notation is used in other categories, for example, if $\mathcal{C}, \mathcal{D}$ are additive categories (with $\mathcal{C}$ skeletally small), then $(\mathcal{C}, \mathcal{D})$ denotes the category of additive functors from $\mathcal{C}$ to $\mathcal{D}$.

$R[X_1, \ldots, X_n]$ denotes the ring of polynomials with coefficients from R in indeterminates $X_1, \ldots, X_n$ which commute with each other and with all the elements of R; $R\langle X_1, \ldots, X_n\rangle$ denotes the free associative R-algebra in indeterminates $X_1, \ldots, X_n$; "non-commutative polynomials over R" with the elements of R acting centrally (R will be commutative whenever this notation appears).

Annihilators: $\mathrm{ann}_R(A) = \{r \in R : ar = 0\ \forall a \in A\}$, where $A \subseteq M_R$, and the obvious modifications $\mathrm{ann}_R(a)$, $\mathrm{ann}_R(a_1, \ldots, a_n)$; $\mathrm{ann}_M(T) = \{a \in M : at = 0\ \forall t \in T\}$ where $T \subseteq R$.

The **socle**, $\mathrm{soc}(M)$, of a module M is the sum of all its simple submodules.

$C(R)$ denotes the **centre**, $\{r \in R : sr = rs\ \forall s \in R\}$, of the ring R.

pp_R^n is the lattice of equivalence classes of pp conditions in n free variables for right R-modules.

$\mathrm{pp}^n(M)$ is the lattice of subgroups of M^n pp-definable in M; this may be identified with pp_R^n modulo the equivalence relation of having the same value on M.

$\mathrm{pp}^n(\mathcal{X})$ is pp_R^n modulo the equivalence relation of having the same value on every member of $\mathcal{X}$ and this is equal to $\mathrm{pp}^n(M)$ for any M with $\langle M\rangle = \mathcal{X}$.

$\mathrm{pp}^n(X)$ (for X a closed subset of the Ziegler spectrum) is $\mathrm{pp}^n(\mathcal{X})$ if $\mathcal{X}$ is a definable subcategory with $\mathcal{X} \cap \mathrm{Zg}_R = X$.

$\mathbb{L}_R^{\mathrm{eq}+}$, $(\mathbb{L}_R^{\mathrm{eq}+})_M$, $(\mathbb{L}_R^{\mathrm{eq}+})_{\mathcal{X}}$ (also $\mathbb{L}^{\mathrm{eq}+}(\mathcal{X})$), $(\mathbb{L}_R^{\mathrm{eq}+})_X$ denote the categories of pp sorts and pp-definable maps for R-modules and then relativised respectively as above.

$\langle M\rangle$ is the definable subcategory generated by M.

$[\phi]_M$ denotes the the image of $\phi \in \mathrm{pp}_R^n$ in $\mathrm{pp}^n(M)$ (similarly with X or $\mathcal{X}$ in place of M).

$[\phi, \psi]$, $[\phi, \psi]_M$, $[\phi, \psi]_{\mathcal{X}}$, $[\phi, \psi]_X$ (with the assumption $\phi \geq \psi$) denote the interval in the lattice pp_R^n between ϕ and ψ and then the respective relativisations as above.

$\mathrm{Zg}(\mathcal{X})$ is $\mathcal{X} \cap \mathrm{Zg}_R$, equipped with the relative topology, when $\mathcal{X}$ is a definable subcategory of Mod-R.

Latt$(-)$ denotes the lattice of subobjects of $(-)$; the lattice of finitely presented, respectively finitely generated, subobjects is indicated by superscript $^{\mathrm{fg}}$, resp. $^{\mathrm{fp}}$, or, if finitely generated = finitely presented, just by $^{\mathrm{f}}$.

Mod(T) is the class of modules on which every pp-pair in T (a set of pp-pairs) is closed.

pinj_R denotes the class of indecomposable pure-injective right R-modules or, more usually, the set of isomorphism classes of these.

Pinj_R denotes the class (or full subcategory) of all pure-injective right R-modules.

proj-R denotes the class of finitely generated projective right R-modules.

Flat-R denotes the class of all flat right R-modules and R-Flat the class of all flat left R-modules; and similarly, lower case indicates "small" (indecomposable, finitely presented, finitely generated, as appropriate), upper case indicates all.

The choice of subscript $_R$ or notation like -R is not significant (thus $\mathrm{Inj}_R = \mathrm{Inj}\text{-}R = \mathrm{Inj}(\mathrm{Mod}\text{-}R)$). What is significant is the notational difference between proj-$\mathcal{A}$ (or $\mathrm{proj}_{\mathcal{A}}$) and proj($\mathcal{A}$): the former refers to projective right modules *over* the category $\mathcal{A}$, the latter to projective objects *in* the category $\mathcal{A}$.

I will write, for instance, "the set of indecomposable injectives" even though it is the isomorphism types of these which form a set and which is what I really mean.

These are various ways of writing the same thing: $\overline{a} \in \phi(M)$; $M \models \phi(\overline{a})$; $\overline{a}$ satisfies ϕ in M; $\phi(\overline{a})$ is true in M; $\overline{a}$ is a solution of/to ϕ in M. And similarly with a pp-type p in place of the pp condition ϕ.

$N|M$ means N is a direct summand of M.

Add($\mathcal{Y}$), respectively add($\mathcal{Y}$), denotes the closure of the class $\mathcal{Y}$ under direct summands of arbitrary, respectively finite, direct sums.

Here are four notations for the same object: $\mathbb{L}_R^{\mathrm{eq}+}$; $(\mathrm{mod}\text{-}R, \mathbf{Ab})^{\mathrm{fp}}$; $\mathrm{Ab}(R^{\mathrm{op}})$; fun-$R$. Defined as different objects, these small abelian categories turn out to the same (that is, equivalent categories).

Selected notations list

supp(M): support of M
$\mathrm{pp}^M(\overline{a})$: pp-type of $\overline{a}$ in M
$H(M)$: pure-injective hull of M
$H(\overline{a})$: hull of $\overline{a}$
$H(p)$: hull of pp-type p
$E(M)$: injective hull of M
$\langle M\rangle$: definable subcategory generated by M
$\langle\phi\rangle$: pp-type generated by ϕ
$D\phi$: (elementary) dual of pp condition ϕ
dF: dual of functor F
$(M, \overline{a})$: pointed module
$\phi(M)$: solution set of ϕ in M
R_M: ring of definable scalars of M
$(A, -)$: representable functor
$(\star)_{(-)}$: localisation of $\star$ at $-$
(ϕ/ψ), (F), (f): basic Ziegler-open sets
$[\phi/\psi]$ etc.: complement, $(\phi/\psi)^c$, of above
$(-)^{\mathrm{fp}}$: subcategory of finitely presented presented objects
$\overrightarrow{F}$: extension of F to functor commuting with direct limits
$\mathbb{L}_R^{\mathrm{eq}+}$: category of imaginaries/pp sorts and maps
Zg_R, $\mathrm{Zg}(\mathcal{D})$: Ziegler spectrum
Zar_R: rep-Zariski spectrum
(L)Def_R: sheaf/presheaf of definable scalars
pp_R^n: lattice of pp conditions
$\mathrm{Latt}^{\mathrm{f}}$: lattice of “finite” subobjects
fun-R and $\mathrm{fun}^{\mathrm{d}}$-$R$; functor category and dual functor category
w($-$): width
mdim($-$); m-dimension
KGdim($-$), KG($-$): Krull–Gabriel dimension
Udim($-$), UD($-$): uniserial dimension
CB($-$): Cantor–Bendixson rank