ROTATING RELATIVISTIC STARS

The masses of neutron stars are limited by an instability to gravitational collapse, and an instability driven by gravitational waves limits their spin. Their oscillations are relevant to X-ray observations of accreting binaries and to gravitational wave observations of neutron stars formed during the coalescence of double neutron-star systems. This volume pulls together more than forty years of research to provide graduate students and researchers in astrophysics, gravitational physics, and astronomy with the first self-contained treatment of the structure, stability, and oscillations of rotating neutron stars. This monograph treats the equations of stellar equilibrium; key approximations, including slow rotation and perturbations of spherical and rotating stars; stability theory and its applications, from convective stability to the r-mode instability; and numerical methods for computing equilibrium configurations and the nonlinear evolution of their oscillations. The presentation of fundamental equations, results, and applications is accessible to readers who do not need the detailed derivations.

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Rotating Relativistic Stars

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To Paula

To my parents, Christoforos and Evangelia

Contents

Preface

The masses of neutron stars are limited by an instability to collapse, and an instability driven by gravitational waves may limit their spin. Their oscillations are relevant to X-ray observations of accreting binaries and to gravitational wave observations of neutron stars formed during the coalescence of double neutron-star systems. This volume pulls together more than 40 years of research to provide graduate students and researchers in astrophysics, gravitational physics, and astronomy with a self-contained treatment of the structure, stability, and oscillations of rotating relativistic stars. Numerical and analytic work are both essential to the subject, and their interplay is emphasized in our treatment.

The book is intended for more than one audience: Readers who need to work through mathematical details of stellar perturbations and stability theory will find them here, in derivations and proofs of principal results. More commonly, a reader working in relativistic astrophysics will want the principal results of the theory but will need only a few of the derivations. The text is also designed to provide a coherent treatment for this second audience, with an exposition of the results preceding the more mathematical derivations. Although our primary concern is with rotating stars, we begin our discussion of oscillations and stability with spherical stars for completeness and to make the presentation accessible to readers with no previous knowledge of relativistic perturbation theory.

Those intending to work through the mathematical derivations should have a background comparable to a semester of gravitational physics at the level of MTW [480] or Wald [715]. The rest of the book should be accessible to students who have mastered Schutz's *First Course in General Relativity* [595], supplemented by the appendices here on Lie derivatives and integration.

We are indebted to all our long-term collaborators in a large number of joint publications, the main results of which are presented in this book. We are especially grateful for their contributions in several research areas that are presented here in an abridged form.

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List of symbols

This is a global glossary, restricted to symbols used in more than one place in the text. Local uses of symbols that appear within a page or two of their definition are in general not listed here. For example, h is used globally to mean specific relativistic enthalpy, and that definition is listed here; h is also used on a single page to mean the amplitude of a gravitational wave, and that local definition is not listed.

A	a generic constant
$A \ \ (A_\alpha)$	Schwarzschild discriminant (vector version)
A_α	electromagnetic vector potential
A_{ab}	trace-free part of extrinsic curvature
$\tilde{A}_{ab}$	related by conformal factor to A_{ab}
$\mathcal{A}, \mathcal{A}^a$	generic densities
a	constant in asymptotic metric of rotating star
	J/M in the Kerr geometry
B	metric potential of rotating star
B^α	magnetic field
$\mathcal{B}$	bag constant in quark interactions
b	constant in asymptotic form of metric
C	generic constant
C_μ	gravitational constraint
c	speed of light
$c_{\rm s}$	speed of sound
$c(\tau), c(\lambda)$	path in spacetime
D_{lm}	mass multipole moment
D_a	covariant derivative of spatial metric γ_{ab}
$\tilde{D}_a$	covariant derivative of spatial metric $\tilde{\gamma}_{ab}$
d	exterior derivative
$d\sigma_\alpha, d\sigma_a$	$dS_\alpha/\sqrt{\lvert g\rvert}, dS_a/\sqrt{\gamma}$
dl	element of proper length
dS_α, dS_a	elements of (hyper)surface area
dV, d^4V	3- and 4-dimensional volume elements
E	energy
E_c	canonical energy
$E_{c,r}$	canonical energy in rotating frame
$E_{\alpha\beta}$	$G_{\alpha\beta} - 8\pi T_{\alpha\beta}$

E_{ab}	spatial projection of $E_{\alpha\beta}$
E^α	electric field
$\mathcal{E}$	injection energy
e	specific internal energy (per unit baryon mass)
$\mathbf{e}_{\hat{\mu}}$	contravariant basis vector of ZAMO
e_{ab}	metric on a sphere of radius r
F	the scalar $u^t u_\phi$
	a generic function
$\mathcal{F}$	function in Eulerian perturbation theory
$F^{\alpha\beta}$	electromagnetic field tensor (Faraday tensor)
f	a generic function
f_α	4-force per unit volume
G	Newton's constant
	occasionally a generic function
$G_{\alpha\beta}$	Einstein tensor
$G_{\alpha\beta\gamma\delta}$	metric expression appearing in $\delta(G^{\alpha\beta}\sqrt{\lvert g\rvert})$
g	specific Gibbs free energy
	determinant of spacetime metric
$g_{\alpha\beta}$	spacetime metric
H	thermodynamic quantity, $\ln h$ for homentropic fluid
	metric potential in slow-rotation approximation
$\mathcal{H}$	Hamiltonian density
H_0, H_1, H_2	potentials of polar metric perturbation of spherical star
h	specific enthalpy (per unit baryon mass)
$h_{\alpha\beta}$	metric perturbation: $\delta g_{\alpha\beta}$
h_0, h_1, h_2	potentials of axial metric perturbation of spherical star
I	moment of inertia
	action
$\mathcal{I}$	null infinity (scri)
J	angular momentum
	Jacobian of a diffeomorphism
J_c	canonical angular momentum
$J_{c,r}$	canonical angular momentum in rotating frame
J_{lm}	current multipole moment
J^α	generic current
j	specific angular momentum of fluid
	J/M^2, a dimensionless measure of angular momentum
j	$\rho\sqrt{\lvert g\rvert}u^t$
j_a	momentum density of fluid
	conduction current for heat flow
j^α	baryon current density
K	polytropic constant
	trace $K_a{}^a$ of extrinsic curvature
K_{ab}	extrinsic curvature

k	generic constant
	potential of a perturbed metric
k^α	helical Killing vector: $t^\alpha + \Omega\phi^\alpha$
L	angular momentum of free particle
	linear operator in perturbation theory
$\mathcal{L}$	Lie-derivative operator
$\boldsymbol{\mathcal{L}}$	the operator $\lvert g\rvert^{-1/2}\mathcal{L}\,\lvert g\rvert^{1/2}$
$\mathscr{L}$	Lagrangian density
l	label of rotation group representation, as in Y_{lm}
	proper length
M	gravitational mass
$M_{\rm ADM}$	ADM mass
$M_{\rm K}$	Komar mass
M_0	baryon mass of star
$\mathcal{M}$	manifold of fluid trajectories
m	angular eigenvalue, as in $e^{im\phi}$
$m(r)$	mass within Schwarzschild radial coordinate r
m_B	fiducial baryon mass: mass per nucleon of ^{12}C
m_e, m_s	electron mass, strange quark mass
m_n	complex multipole moments
m_0, m_2	potentials of a slowly rotating star
N	polytropic index
	baryon number
N	Brunt-Väisälä frequency
n	baryon number density
	neutron
	generic integer
n_e	electron number density
n^α	future-pointing unit normal to hypersurface
O	an order symbol
o	an order symbol
P	point of spacetime
	period of rotation
	point along a sequence of stars
$\mathcal{P}$	a parity transformation (diffeomorphism)
p	pressure
Q	scalar quadrupole moment
$\mathcal{Q}$	set of variables of perfect-fluid spacetime
Q_{ab}	tensor in Newtonian perturbation theory
q^α	heat-flow vector
$q_\alpha{}^\beta$	projection tensor orthogonal to u^α
R	circumferential equatorial radius of star
	3-dimensional Ricci scalar
${}^4R$	4-dimensional Ricci scalar

$R_{\alpha\beta}, R_{ab}$	4- and 3-dimensional Ricci tensors
$\tilde{R}_{ab}$	Ricci tensor of $\tilde{\gamma}_{ab}$
r	radial coordinate
$\mathbf{r}$	position vector (Newtonian)
r_c	circumferential radial coordinate
S	2-dimensional surface
S^{ab}	stress tensor: spatial projection of $T^{\alpha\beta}$
s	specific entropy (per unit baryon mass)
s^α	entropy current
T	temperature
	rotational kinetic energy of rotating star
T_s	superfluid transition temperature
$T^{\alpha\beta}$	stress-energy tensor
$T^{a_1\cdots a_m}{}_{b_1\cdots b_n}$	a generic tensor
$\mathcal{T}$	a subspace of the space of trivial displacements
t	time coordinate
t^α	time-translation symmetry vector
U	effective potential in two-potential formalism
$U^{\alpha\beta\gamma\delta}$	tensor in Lagrangian perturbation theory
u	null coordinate
u^α	4-velocity of fluid or particle
$V^{\alpha\beta\gamma\delta}$	tensor in Lagrangian perturbation theory
V	3-dimensional region
v	magnitude of 3-velocity measured by a ZAMO
	null coordinate
$\mathbf{v}$, v^a	Newtonian fluid velocity
v^α	spatial part of fluid velocity u^α
	generic vector
W	gravitational potential energy of star
	symplectic structure (form) of perturbation
	Lorentz factor
$W^{\alpha\beta\gamma\delta}$	tensor in expression for $\Delta(T^{\alpha\beta}\sqrt{\lvert g\rvert})$
w	function occurring in Manko formalism
	function occurring in Eulerian perturbation theory
w^α	unnormalized tangent vector to fluid trajectory
	the vector hu^α
	generic vector
X	generic variable
x	coordinate
x^μ	spacetime coordinate
x_p	proton fraction
Y_k	fractional number density of kth species of particle
y	function in Eulerian perturbation theory
	coordinate

Z	Zerilli function
	redshift
z	coordinate
α	lapse
	imaginary part of mode frequency
β^a	shift vector
Γ	$d\log p/d\log\rho$; polytropic exponent
Γ_1	adiabatic index
$\Gamma^\lambda{}_{\mu\nu}$	Christoffel symbol
γ	determinant of 3-metric
	Lorentz factor $1/\sqrt{1-v^2}$
γ_{ab}	3-metric
$\tilde{\gamma}_{ab}$	conformal 3-metric
$\gamma_a{}^\alpha$	pullback of vectors on M to vectors on Σ
Δ	Lagrangian change
$\boldsymbol{\Delta}$	the operator $\lvert g\rvert^{-1/2}\Delta\lvert g\rvert^{1/2}$
δ	Eulerian change
$\boldsymbol{\delta}$	the operator $\lvert g\rvert^{-1/2}\delta\lvert g\rvert^{1/2}$
ϵ, ϵ_c	energy density, central energy density
$\epsilon_{\alpha\beta\gamma\delta}, \epsilon_{abc}$	normalized totally antisymmetric tensors
ϵ_{ab}	normalized antisymmetric tensor on sphere
ζ	metric potential of rotating star
$\boldsymbol{\zeta}$	Newtonian vorticity vector
ζ^α, ζ^a	generator of gauge transformation
ζ^a	generic vector field
η	coefficient of viscosity
$\eta_{\alpha\beta}$	flat Minkowski metric
η_{ab}	flat spatial metric
η^α	trivial Lagrangian displacement
Θ	step function
$\boldsymbol{\Theta}^\alpha$	vector density from variation of action
θ	spatial divergence of u^α
	angular coordinate
κ	coefficient of heat conductivity
	a generic constant
Λ	the Λ hyperon
λ	metric potential of spherical star
	parameter along sequence of stars
μ	metric potential of rotating star
ν	metric potential of rotating or spherical star
ξ^a	Lagrangian displacement
Π	the internal energy of a Newtonian star
Π_α	momentum conjugate to ξ^α in perturbed fluid

Π_a	momentum density of fluid
π	trace, $\pi_a{}^a$
$\pi^{\alpha\beta}$	momentum density conjugate to perturbed metric $h_{\alpha\beta}$
π^{ab}	momentum density conjugate to spatial metric γ_{ab}
$\pi^0, \pi^\pm$	pions
ϖ	cylindrical coordinate radius
ρ	baryon mass density
Σ	3-dimensional hypersurface
	the Σ hyperon
σ	the complex frequency of a mode
$\sigma_{\alpha\beta}$	shear tensor
τ	proper time along a trajectory
$\tau_{\rm b}$	bulk-viscosity damping time
τ_{GW}	gravitational wave damping time or growth time
$\tau_{\rm s}$	shear-viscosity damping time
Υ	scalar occurring in expression for fluid velocity u^α
Φ	metric potential of a spherical star
	Newtonian gravitational potential
Φ^I	one of a set $\{\phi^I\}$ of fields on spacetime
ϕ	angular coordinate
	potential in conformal factor $e^{4\phi}$ of 3-metric
ϕ^α	rotational symmetry vector
$\chi, \chi_{\sf s}$	diffeomorphisms describing fluid configuration
Ψ	velocity potential of irrotational fluid
ψ	diffeomorphism
	metric potential of rotating star
Ω	angular velocity of rotating star
Ω_K	Keplerian angular velocity
ω	frame-dragging potential of rotating star
$\bar\omega$	$\omega - \Omega$, used in slow-rotation formalism
$\omega_{\alpha\beta}$	vorticity tensor
ω_i	real mode frequency measured by inertial observer
ω_r	real mode frequency measured by rotating observer
ω^α	vorticity vector
$\boldsymbol{\omega}^{\hat\mu}$	covariant basis vector of ZAMO
∇_α	covariant derivative operator of spacetime metric $g_{\alpha\beta}$
$\mathit{\nabla}_\alpha$	covariant derivative operator of flat metric $\eta_{\alpha\beta}$
$\mathit{\nabla}_a, \boldsymbol{\nabla}$	covariant derivative operator of flat spatial metric η_{ab}
$\boldsymbol{\partial}_\mu$	contravariant basis vector associated with coordinate x^μ

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Frontmatter
More Information

Conventions, notation, and mathematical preliminaries

Units, metric and physical constants

Throughout the book, gravitational units, with $G = c = 1$, will be adopted in writing the equations governing stellar structure and dynamics, whereas numerical properties of stellar models will be listed in cgs units, unless otherwise noted. We use the conventions of Misner, Thorne, and Wheeler [480] for the signature of the spacetime metric $(-+++)$ and for signs of the curvature tensor and its contractions. Spacetime indices will be Greek, α, $\beta, \ldots$, whereas spatial indices will be Latin $a, b, \ldots$. (Readers familiar with abstract indices can regard indices early in the alphabet as abstract, whereas indices μ, ν, λ and i, j, k will be concrete, taking values $\mu = 0, 1, 2, 3,\ i = 1, 2, 3$.) Corresponding to a choice of coordinates, t, r, θ, ϕ, a vector u^α has components $u^t, \ldots, u^\phi$; its components along an orthonormal frame, $\{\mathbf{e}_{\hat{0}}, \ldots, \mathbf{e}_{\hat{3}}\}$, will be written $\{u^{\hat{0}}, \ldots, u^{\hat{3}}\}$. Parentheses enclosing a set of indices indicate symmetrization, and square brackets indicate antisymmetrization.

Numbers that rely on physical constants are based on the values $c = 2.9979 \times 10^{10}$ cm s^{-1}, $G = 6.670 \times 10^{-8}$ g^{-1}cm^{3}s^{-2}, $\hbar = 1.0545 \times 10^{-27}$ g cm^{2}s^{-1}, baryon mass $m_B = 1.659 \times 10^{-24}$ g, and $M_\odot = 1.989 \times 10^{33}$ g $= 1.477$ km.

Derivatives and integrals

The covariant derivative operator of the spacetime metric $g_{\alpha\beta}$ will be written ∇_α, and the partial derivative of a scalar f with respect to one of the coordinates – say r – will be written $\partial_r f$ or $f_{,r}$. Lie derivatives along a vector u^α will be denoted by $\mathcal{L}_{\mathbf{u}}$. The Lie derivative of an arbitrary tensor $T^{a\cdots b}{}_{c\cdots d}$ is

$$\mathcal{L}_{\mathbf{u}} T^{a\cdots b}{}_{c\cdots d} = u^e \nabla_e T^{a\cdots b}{}_{c\cdots d} - T^{e\cdots b}{}_{c\cdots d} \nabla_e u^a - \cdots - T^{a\cdots e}{}_{c\cdots d} \nabla_e u^b + T^{a\cdots b}{}_{e\cdots d} \nabla_c u^e + \cdots + T^{a\cdots b}{}_{c\cdots e} \nabla_d u^e. \tag{1}$$

Our notation for integrals is as follows. We denote by d^4V the spacetime volume element. In a chart $\{x^0, x^1, x^2, x^3\}$, the notation means

$$d^4V = \epsilon_{0123} dx^0 dx^1 dx^2 dx^3 = \sqrt{|g|}\, d^4x, \tag{2}$$

where g is the determinant of the matrix $\|g_{\mu\nu}\|$. Gauss's theorem (presented in Section A.3 of the Appendix) has the form

$$\int_\Omega \nabla_\alpha A^\alpha d^4V = \int_{\partial\Omega} A^\alpha dS_\alpha, \tag{3}$$

with $\partial\Omega$ the boundary of the region Ω. In a chart (u, x^1, x^2, x^3) for which V is a surface of constant u, $dS_\alpha = \sqrt{|g|}\,\nabla_\alpha u d^3x$, and

$$\int_V A^\alpha dS_\alpha = \int_V A^u \sqrt{|g|} d^3x. \tag{4}$$

If V is nowhere null, one can define a unit normal,

$$\widehat{n}_\alpha = \frac{\nabla_\alpha u}{\left|\nabla_\beta u \nabla^\beta u\right|^{1/2}}\,, \tag{5}$$

and write

$$dS_\alpha = \widehat{n}_\alpha dV, \tag{6}$$

where

$$dV = \sqrt{|^3g|}\; d^3x, \tag{7}$$

where ${}^3g$ is the determinant of the 3-metric induced on the surface V. But Gauss's theorem has the form (3) for any 3-surface S, bounding a 4-dimensional region $\mathcal{R}$, regardless of whether S is timelike, spacelike, or null.[1]

Similarly, if $F^{\alpha\beta}$ is an antisymmetric tensor, its integral over a 2-surface S of constant coordinates u and v is written

$$\int_S F^{\alpha\beta} dS_{\alpha\beta} = \int_S F^{uv} \sqrt{|g|} d^2x, \tag{8}$$

and a corresponding generalized Gauss's theorem has the form

$$\int_V \nabla_\beta F^{\alpha\beta} dS_\alpha = \int_{\partial V} F^{\alpha\beta} dS_{\alpha\beta}. \tag{9}$$

If n_α and $\widetilde{n}_\alpha$ are orthogonal unit normals to the surface S, for which $(\mathbf{n}, \widetilde{\mathbf{n}}, \boldsymbol{\partial}_2, \boldsymbol{\partial}_3)$ is positively oriented, then $dS_{\alpha\beta} = n_{[\alpha}\widetilde{n}_{\beta]} \sqrt{|^2g|}\; d^2x$.

Asymptotic notation: O and o

We will use the symbols $O(x)$ and $o(x)$ to describe asymptotic behavior of functions. For a function $f(x)$, $f = O(x)$ if there is a constant C for which $|f/x| < C$, for sufficiently small $|x|$; $f = o(x)$ if $\lim_{x\to 0} |f/x| = 0$. For example, if A is constant, $A/r = O(r^{-1})$, and $A/r^{3/2} = o(r^{-1})$.

[1] Note that in the text, n_α denotes the *future* pointing unit normal to a $t =$ constant hypersurface, $n_\alpha = -\nabla_\alpha t / |\nabla_\beta t \nabla^\beta t|^{1/2}$. In order that, for example, $\int \rho u^\alpha dS_\alpha$, be positive on a $t =$ constant surface, one must use $dS_\alpha = \nabla_\alpha t \sqrt{|g|} d^3x = \widehat{n}_\alpha dV = -n_\alpha dV$.