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Stationary, axisymmetric equilibria

The Newtonian approximation describes to extraordinarily high precision the gravitational field of low-mass stars over the course of their evolution, from the instability to collapse that triggers their formation to their death as white dwarfs. In a high-mass star, however, when the nuclear reactions that halted its initial collapse ultimately die out, the core's renewed collapse leads either to a star above nuclear density or to a black hole at whose center is a speck that, at least momentarily, is vastly beyond any known density. In both of these final states of stellar evolution, general relativity plays a fundamental role. The relativistic stars of nature have a complex composition, spanning fifteen orders of magnitude in density. Thought to consist primarily of a gas of neutrons with a gradually varying density of free protons, electrons, and muons, they are surrounded by a crust of ordinary matter, and their cores may hold hyperons, pion or kaon condensates, or possibly free quarks. In fact, our uncertainty about the behavior of matter above nuclear density is (in 2012) great enough to allow what we call neutron stars to be strange-quark stars, collections of up, down, and strange quarks surrounded by a thin normal crust. In the conventional neutron-star model, a much thicker, 1-km crust surrounds an interior in which neutrons and protons form a two-component superfluid. High magnetic fields, whose strength in some cases appears to exceed 10^{14} G, are observed and thought to extend in quantized flux tubes through the superfluid interior. The angular velocities of observed millisecond pulsars range up to 716 Hz, and the vorticity of their velocity fields is similarly thought to be confined, in the neutron stars' interiors, to quantized tubes.

Departures from local isotropy are associated with the crust, with the vortex and magnetic flux tubes, and with heat flow and viscosity. Nevertheless, a neutron star *in equilibrium* is accurately approximated by a stationary self-gravitating perfect fluid, its structure determined by a balance among its intense gravity, the pressure of its degenerate particles, and its rotation. In particular, departures from perfect fluid equilibrium due to a solid crust are expected to be smaller than $\sim 10^{-3}$, corresponding to the maximum strain that an electromagnetic lattice can support [142]; this estimate is supported by observations of pulsar glitches, which are consistent with departures from a perfect fluid equilibrium of order 10^{-5} (see [235]).

Similarly, on scales of meters or larger, a single rotational velocity field u^α describes the averaged superfluid motion [58, 639, 424]. The error in computing the gravitational field is much smaller than errors in the fluid model, because the characteristic length over which a potential varies is much larger than the distance between vortices. Although the assumption of a perfect fluid is adequate for describing equilibrium configurations, studies of neutron-star dynamics – of formation, oscillations, and stability, and of the interaction of binaries during and just before merger – require a more detailed knowledge of the stars' microphysics.

1.1 Perfect fluids

The stress-energy tensor. A perfect fluid is a model for a large assembly of particles in which a continuous energy density ϵ can reasonably describe the macroscopic distribution of mass. One assumes that the microscopic particles collide frequently enough that their mean free path is short compared with the scale on which the density changes, so that the collisions enforce a local thermodynamic equilibrium. In particular, one assumes that a mean velocity field u^α and a mean stress-energy tensor $T^{\alpha\beta}$ can be defined in boxes – fluid elements – that are small compared to the macroscopic length scale but large compared to the mean free path. One also assumes that on scales large compared to the size of the fluid elements, the 4-velocity and thermodynamic quantities can be accurately described by continuous fields. An observer moving with the average velocity u^α of the fluid will see the collisions randomly distribute the nearby particle velocities so that the particle distribution will appear locally isotropic.

Because a comoving observer sees an isotropic distribution of particles, the components of the fluid's energy momentum tensor in her frame must have no preferred direction: $T^{\alpha\beta}u_\beta$ must be invariant under rotations that fix u^α. Denote by

$$q^{\alpha\beta} = g^{\alpha\beta} + u^\alpha u^\beta \tag{1.1}$$

the projection operator orthogonal to u^α. The momentum current

$$q^\alpha{}_\gamma T^{\gamma\beta}u_\beta$$

is a vector in the 3-dimensional subspace orthogonal to u^α, and it is therefore invariant under rotations of that subspace only if it vanishes. Similarly, the symmetric trace-free tensor ${}^3T^{\alpha\beta} - \frac{1}{3}q^{\alpha\beta}\,{}^3T \equiv q^\alpha{}_\gamma q^\beta{}_\delta T^{\gamma\delta} - \frac{1}{3}q^{\alpha\beta}q_{\gamma\delta}T^{\gamma\delta}$ belongs to a $j = 2$ representation of the rotation group and can be invariant only if it vanishes.

It follows that the only nonzero parts of $T^{\alpha\beta}$ are the rotational scalars

$$\epsilon := T^{\alpha\beta}u_\alpha u_\beta \tag{1.2}$$

and

$$p := \frac{1}{3}q_{\gamma\delta}T^{\gamma\delta}. \tag{1.3}$$

More concretely, in an orthonormal frame $\mathbf{e}_{\hat\mu}$, with $e^\alpha_{\hat 0} = u^\alpha$, the components $T^{\hat 0 \hat\imath}$ and $T^{\hat\imath\hat\jmath} - \frac{1}{3}\delta^{\hat\imath\hat\jmath}T^{\hat k}{}_{\hat k}$ must vanish, implying that $T^{\alpha\beta}$ has components

$$\|T^{\hat\mu\hat\nu}\| = \begin{Vmatrix} \epsilon & & & \\ & p & & \\ & & p & \\ & & & p \end{Vmatrix}. \tag{1.4}$$

To summarize: The condition of local isotropy suffices to define a perfect fluid by enforcing a stress-energy tensor (synonymous with *energy-momentum tensor*) of the form

$$T^{\alpha\beta} = \epsilon u^\alpha u^\beta + p q^{\alpha\beta}. \tag{1.5}$$

The scalars ϵ and p are the *total energy density* (or, simply, energy density) and the *pressure*, as measured by a comoving observer (an observer with 4-velocity u^α).

Thermodynamics. We denote by n the baryon number density and assign a fixed rest mass m_B per baryon.[1] The rest-mass density (equivalently, baryon-mass density) is then

$$\rho := m_B n. \tag{1.6}$$

In general, the properties of matter in a compact object will depend on several parameters, including fluid and magnetic stresses, entropy gradients, composition, heat flow, and neutrino emission. Here, we restrict our attention to the case of a perfect fluid with equilibrium composition, where the energy density and pressure depend on two parameters that can be taken to be ρ and the *specific entropy* (entropy per unit rest mass) s,

$$\epsilon = \epsilon(\rho, s), \qquad\qquad p = p(\rho, s). \tag{1.7}$$

The thermodynamics of the fluid is described by the first law, which, in terms of ρ and s, takes the form

$$d\epsilon = \rho T ds + h d\rho, \tag{1.8}$$

where T is *temperature* and h is the *specific enthalpy* (enthalpy per unit rest mass)

$$h := \frac{\epsilon + p}{\rho}. \tag{1.9}$$

One can easily derive Eq. (1.8) from its more common form in terms of extensive quantities,

$$dE = TdS - p\,dV + \mu\,dN \equiv TdS - p\,dV + g\,dM_0, \tag{1.10}$$

[1] Assignment of a rest mass density is somewhat arbitrary, but the difference between choices is less than 0.1%. We follow earlier papers in using the mass per nucleon of ^{12}C, $m_B = 1.659 \times 10^{-24}$ g. This choice is equivalent, up to a constant factor, to assigning to a fluid the rest mass of the collection of free electrons and protons that would result from dispersing the fluid.

by introducing the energy E, entropy S, volume V, baryon number N, and rest mass $M_0 = m_B N$ of a fluid element as measured by a comoving observer. (Here $\mu = g m_B$, where the quantity g is to be defined below.) Then, writing

$$\epsilon = \frac{E}{V}, \quad s = \frac{S}{M_0}, \quad \rho = \frac{M_0}{V},$$

and using conservation of baryons in the form $dM_0 = 0$ to replace dV/V by $-d\rho/\rho$, we quickly obtain Eq. (1.8).

Note that, whether or not one adds baryons to the fluid, one can choose dV so that the number of baryons in the volume $V + dV$ is unchanged; Eq. (1.8) thus holds whether or not baryons are added to the fluid. On the other hand, if one simply enlarges the volume under consideration without changing the internal state of the fluid (keeping ρ and s constant), then $d\rho = ds = d\epsilon = 0$, $dM_0 = \rho dV$, and the extensive version (1.10) of the first law implies

$$\epsilon dV = d(\epsilon V) = T d(\rho s V) - p\, dV + g\, d(\rho V) = (\rho T s - p + \rho g) dV,$$

whence

$$g = \frac{\epsilon + p}{\rho} - Ts. \tag{1.11}$$

The quantity g is thus the *specific Gibbs free energy* (free energy per unit rest mass), and $\mu = g m_B = \dfrac{\epsilon + p - \rho T s}{n}$ is the *Gibbs free energy per baryon*, each defined for a comoving observer.

Defining the *specific internal energy* (internal energy per unit rest mass) e by the relation

$$\epsilon = \rho(1 + e), \tag{1.12}$$

one recovers the *Newtonian expression for the specific enthalpy*,

$$h_{\text{Newtonian}} = h - 1 = e + p/\rho. \tag{1.13}$$

Because the relativistic energy density ϵ includes the rest-mass density ρ, the relativistic enthalpy per unit rest mass differs from its Newtonian counterpart by the rest mass per unit rest mass, by $\rho/\rho = 1$.

Baroclinic (entropy nonconserving) flow. From the definition (1.9) and using Eq. (1.8), one finds

$$\begin{aligned} dh &= \frac{d\epsilon}{\rho} + \frac{dp}{\rho} - \frac{\epsilon + p}{\rho^2} d\rho = T ds + h\frac{d\rho}{\rho} + \frac{dp}{\rho} - h\frac{d\rho}{\rho} \\ &= T ds + \frac{dp}{\rho} \\ \Rightarrow \quad d\ln h &= \frac{T}{h} ds + \frac{dp}{\epsilon + p}, \end{aligned} \tag{1.14}$$

implying

$$\nabla_\alpha \ln h = \frac{T}{h}\nabla_\alpha s + \frac{1}{\epsilon + p}\nabla_\alpha p. \tag{1.15}$$

Taking the curl of (1.15), we arrive at

$$0 = \nabla_{[\alpha}\nabla_{\beta]} \ln h = \nabla_{[\alpha}\left(\frac{T}{h}\nabla_{\beta]} s\right) + \nabla_{[\alpha}\left(\frac{1}{\epsilon + p}\nabla_{\beta]} p\right). \tag{1.16}$$

The last relation implies that in the presence of entropy gradients ($\nabla s \neq 0$), surfaces of constant energy density (*isopycnic surfaces*) do not, in general, coincide with surfaces of constant pressure (*isobaric surfaces*). Such a flow is called *baroclinic*, and, for a rotating star, it implies the presence of *meridional circulation*.

Barotropic flow. One commonly uses a one-parameter equation of state to describe a compact star, because within a short time after formation, neutrino emission cools the star to 10^{10} K $\simeq 1$ MeV. This is much smaller than the Fermi energy of the interior, in which a density greater than nuclear ($0.16\,\mathrm{fm}^{-3}$) implies a Fermi energy greater than $E_F\,(0.16\,\mathrm{fm}^{-3}) \approx 60\,\mathrm{MeV}$. A neutron star is in this sense cold, and, because nuclear reaction times are shorter than the cooling time, one can use a zero-temperature *equation of state* (EOS) to describe the matter:

$$\epsilon = \epsilon(\rho), \quad p = p(\rho), \tag{1.17}$$

or, equivalently,

$$\epsilon = \epsilon(p). \tag{1.18}$$

In a stationary, one-component perfect fluid, a one-parameter equation of state of the form (1.18) holds, more generally, when the specific entropy is constant throughout the star ($\nabla s = 0$) – that is, for a *homentropic flow.* From Eq. (1.16) it is evident that in such a case, the isopycnic and isobaric surfaces coincide – that is, the homentropic flow of a one-component perfect fluid is *barotropic*, which is also implied by Eq. (1.18) itself.

In a homentropic star, the first law, Eq. (1.8), takes the form

$$d\epsilon = h d\rho, \tag{1.19}$$

and using Eq. (1.14), the specific enthalpy is also given by

$$h = \exp\left(\int_0^p \frac{dp}{\epsilon + p}\right), \tag{1.20}$$

with $\frac{\epsilon}{\rho} = 1$ at $p = 0$ (the gas is nonrelativistic at low densities).

Although a nonhomentropic star is, in general, a barocline, if one makes the assumption that $\epsilon = \epsilon(p)$, then Eq. (1.16) implies, for a one-component perfect fluid, that $s = s(\rho)$, and the star is still barotropic.[2] Similarly, if $s = s(\rho)$, the star is barotropic.

[2] Such models are sometimes called *pseudobarotropes* in the literature.

Fluid dynamics and conservation laws. For a two-parameter equation of state, five variables determine the state of a perfect fluid; they can be taken to be ϵ, p and three independent components of u^α. The dynamical evolution of the fluid is governed by the vanishing divergence of the stress-energy tensor

$$\nabla_\beta T^{\alpha\beta} = 0, \tag{1.21}$$

and by conservation of baryons,

$$\nabla_\alpha(\rho u^\alpha) = 0. \tag{1.22}$$

The projection of the equation $\nabla_\beta T^{\alpha\beta} = 0$ along u^α yields an energy conservation law, whereas the projection orthogonal to u^α is the relativistic Euler equation. For an intuitive understanding of these equations, it is helpful to look first at conservation of baryons.

Conservation of baryons. The baryon mass M_0 of a fluid element is conserved by the motion of the fluid. The proper volume of a fluid element is the volume V of a slice orthogonal to u^α through the history of the fluid element; conservation of baryons can be written in the form $0 = \Delta M_0 = \Delta(\rho V)$. The fractional change in V in a proper time $\Delta\tau$ is given by the 3-dimensional divergence of the velocity in the subspace orthogonal to u^α:

$$\frac{\Delta V}{V} = q^{\alpha\beta}\nabla_\alpha u_\beta \Delta\tau. \tag{1.23}$$

Because $u^\beta u_\beta = -1$, we have $u^\beta \nabla_\alpha u_\beta = \frac{1}{2}\nabla_\alpha(u_\beta u^\beta) = 0$, implying

$$q^{\alpha\beta}\nabla_\alpha u_\beta = \nabla_\beta u^\beta. \tag{1.24}$$

With $u^\alpha \nabla_\alpha \rho = \dfrac{d}{d\tau}\rho$, conservation of baryons takes the form

$$0 = \frac{\Delta(\rho V)}{V} = \Delta\rho + \rho\frac{\Delta V}{V} = (u^\alpha\nabla_\alpha\rho + \rho\nabla_\alpha u^\alpha)\Delta\tau, \tag{1.25}$$

or

$$\nabla_\alpha(\rho u^\alpha) = 0. \tag{1.26}$$

Conservation of energy. The projection $u_\alpha \nabla_\beta T^{\alpha\beta} = 0$ similarly expresses energy conservation for a fluid element:

$$\begin{aligned} 0 = u_\alpha \nabla_\beta T^{\alpha\beta} &= u_\alpha \nabla_\beta[\epsilon u^\alpha u^\beta + p q^{\alpha\beta}] \\ &= -\nabla_\beta(\epsilon u^\beta) + p u_\alpha \nabla_\beta(g^{\alpha\beta} + u^\alpha u^\beta) \\ &= -\nabla_\beta(\epsilon u^\beta) - p\nabla_\beta u^\beta, \end{aligned}$$

implying

$$\nabla_\beta(\epsilon u^\beta) = -p\nabla_\beta u^\beta. \tag{1.27}$$

The equation means that the total energy of a fluid element decreases by the work,

$$p\, dV = pV\ \nabla_\beta u^\beta\ d\tau, \tag{1.28}$$

that it does on its surroundings in proper time $d\tau$.

Relativistic Euler equation. The projection of the conservation of the stress-energy tensor orthogonal to u^α is

$$q^\alpha{}_\gamma \nabla_\beta T^{\beta\gamma} = 0, \tag{1.29}$$

so that

$$\begin{aligned} 0 &= q^\alpha{}_\gamma \nabla_\beta [\epsilon u^\beta u^\gamma + p q^{\beta\gamma}] \\ &= q^\alpha{}_\gamma \epsilon u^\beta \nabla_\beta u^\gamma + q^{\alpha\beta} \nabla_\beta p + q^\alpha{}_\gamma p \nabla_\beta (u^\beta u^\gamma) \\ &= \epsilon u^\beta \nabla_\beta u^\alpha + q^{\alpha\beta} \nabla_\beta p + p u^\beta \nabla_\beta u^\alpha , \end{aligned}$$

implying

$$(\epsilon + p) u^\beta \nabla_\beta u^\alpha = -q^{\alpha\beta} \nabla_\beta p. \tag{1.30}$$

For a barotropic fluid with constant entropy (a homentropic fluid), one can use Eq. (1.20) to write the relativistic Euler equation in the form

$$u^\beta \nabla_\beta u^\alpha = -q^{\alpha\beta} \nabla_\beta \ln h \tag{1.31}$$

or, equivalently,

$$u^\beta \nabla_{[\alpha} (h u_{\beta]}) = 0. \tag{1.32}$$

In this equation, the form $\omega_{\alpha\beta}$, defined by

$$\omega_{\alpha\beta} = \nabla_\alpha (h\, u_\beta) - \nabla_\beta (h\, u_\alpha), \tag{1.33}$$

is the *relativistic vorticity.*

Newtonian approximation. Let ε be a small parameter of order v/c or $v_{\rm sound}/c$, whichever is larger. In the Newtonian approximation, there are Cartesian coordinates for which the metric has the form

$$ds^2 = -(1 + 2\Phi) dt^2 + (dx^2 + dy^2 + dz^2)(1 + O(\varepsilon^2)), \tag{1.34}$$

and with off-diagonal terms of order $\varepsilon\Phi$. Here, Φ is the Newtonian potential, satisfying

$$\nabla^2 \Phi = 4\pi\rho. \tag{1.35}$$

The components u^μ and scalars Φ, ϵ, p, ρ satisfy relations of the form $\partial_t \Phi = O(\varepsilon)\nabla_i \Phi$ and have the orders

$$u^t = 1 + O(\varepsilon^2), \tag{1.36}$$

$$u^i = v^i + O(\varepsilon^3), \tag{1.37}$$

$$p/\epsilon = O(\varepsilon^2), \tag{1.38}$$

$$\epsilon = \rho[1 + O(\varepsilon^2)]. \tag{1.39}$$

Conservation of baryons (1.26) then takes the form

$$\partial_t \rho + \partial_i(\rho v^i) = 0 + O(\rho\varepsilon^2). \tag{1.40}$$

The relativistic Euler equation becomes

$$\rho u^\mu \nabla_\mu u^i = -\nabla^i p;$$

and, after using the metric to compute $\Gamma^i_{tt} = \nabla_i \Phi[1 + O(\varepsilon^2)]$, we recover the Euler equation

$$\rho(\partial_t + v^j \nabla_j)v_i + \rho\partial_i \Phi = -\nabla_i p. \tag{1.41}$$

Conservation of energy (1.27) to the lowest nontrivial order, $O(\varepsilon^3)$, immediately reduces to conservation of baryons. To $O(\varepsilon^5)$, one obtains the energy conservation equation that arises from the Euler equation (1.41) by dotting it with v^i, but to deduce this relation from Eq. (1.27), one must keep subleading terms in the metric and 4-velocity.

Spacetime symmetries. A vector field ξ^α is a Killing vector if it Lie derives the metric

$$\mathcal{L}_{\boldsymbol{\xi}} g_{\alpha\beta} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0. \tag{1.42}$$

A Killing vector generates a family of isometries, diffeomorphisms that leave the metric invariant. We will call ξ^α a *symmetry vector* of a perfect-fluid spacetime if ξ^α is a Killing vector that also Lie derives the fluid variables:

$$\mathcal{L}_{\boldsymbol{\xi}} u^\alpha = 0, \quad \mathcal{L}_{\boldsymbol{\xi}} \epsilon = 0, \quad \mathcal{L}_{\boldsymbol{\xi}} p = 0. \tag{1.43}$$

Associated with a symmetry ξ^α is a quantity $hu_\beta \xi^\beta$ that is conserved along the spacetime trajectories of the fluid,

$$\mathcal{L}_{\mathbf{u}} \left(h\, u_\beta \xi^\beta\right) = 0. \tag{1.44}$$

To derive this relation, we use the form (1.32) of the Euler equation to write

$$\begin{aligned} 0 &= u^\beta [\nabla_\beta(hu_\alpha) - \nabla_\alpha(hu_\beta)]\, \xi^\alpha \\ &= u^\beta \nabla_\beta(hu_\alpha\, \xi^\alpha) - u^\beta\, \mathcal{L}_\xi(hu_\beta) \\ &= u^\beta \nabla_\beta(hu_\alpha\, \xi^\alpha), \end{aligned}$$

as claimed.

Stationary flow. A stationary flow is described by a spacetime with an asymptotically timelike symmetry vector, t^α, the generator of time translations that leave the metric and the fluid variables fixed. The corresponding conservation law, (1.44),

$$\mathcal{L}_{\mathbf{u}}\left(h\, u_\beta t^\beta\right) = \mathcal{L}_{\mathbf{u}}\left(\frac{\epsilon + p}{\rho} u_\beta t^\beta\right) = 0, \tag{1.45}$$

is the relativistic form of Bernoulli's law, the conservation of *enthalpy per unit rest mass*, $-hu_t$, along the trajectories of a stationary flow.

To obtain the Newtonian approximation,

$$(\partial_t + \mathcal{L}_{\mathbf{v}})\left(h_{\text{Newtonian}} + \frac{1}{2}v^2 + \Phi\right) = 0,$$

one must use the $O(\varepsilon^2)$ form of u_t, namely $u_t = -(1 + \Phi + \frac{1}{2}v^2)$, implied by Eqs. (1.34), (1.36), (1.37) and the normalization $u^\alpha u_\alpha = -1$. Note that the definition $h_{\text{Newtonian}} = h - 1$ coincides for isentropic flows with

$$h_{\text{Newtonian}} = \int_0^p \frac{dp}{\rho}, \tag{1.46}$$

as follows from Eq. (1.13) and the first law in the form $de = \dfrac{p}{\rho^2} d\rho$.

Axisymmetric flow. An axisymmetric flow is described by a spacetime with a rotational symmetry vector ϕ^α, a spacelike vector field whose orbits are circles, except on an axis of symmetry (a two-dimensional submanifold of the spacetime), where $\phi^\alpha = 0$. The corresponding conservation law, (1.44),

$$\mathcal{L}_{\mathbf{u}}(h\, u_\beta \phi^\beta) = 0, \tag{1.47}$$

expresses the conservation of a fluid element's *specific angular momentum*, $j := hu_\phi$, the angular momentum per unit rest mass about the axis of symmetry associated with ϕ^α. We will see in Section 1.8 that calling j the specific angular momentum is consistent with the integral expression for the total angular momentum J of the spacetime: $J = \int j dM_0$.

Isentropic flow. In the absence of shocks, the flow of a perfect fluid remains *isentropic* – that is, each fluid element conserves its specific entropy along its trajectory,

$$u^\alpha \nabla_\alpha s = 0. \tag{1.48}$$

Formally, the relation follows from conservation of baryons (1.26), conservation of energy (1.27), and from the first law (1.8) or equivalently the equation of state $\epsilon = \epsilon(\rho, s)$.

Conservation of vorticity and circulation in barotropic flows. The *relativistic vorticity* $\omega_{\alpha\beta}$ was defined by Eq.(1.33). From Eq. (1.32) and the Cartan identity (A.38), we have

$$0 = u^\beta[\nabla_\beta(hu_\alpha) - \nabla_\alpha(hu_\beta)] = \mathcal{L}_u(hu_\alpha) + \nabla_\alpha h. \tag{1.49}$$

From the fact that exterior derivatives and Lie derivatives commute, $\nabla_{[\alpha}\mathcal{L}_{\mathbf{u}} w_{\beta]} = \mathcal{L}_{\mathbf{u}}\nabla_{[\alpha} w_{\beta]}$, the variation of vorticity along fluid trajectories is given by

$$\mathcal{L}_{\mathbf{u}}\omega_{\alpha\beta} = -2\nabla_{[\alpha}\nabla_{\beta]}h = 0. \tag{1.50}$$

For a barotropic flow, it immediately follows that *vorticity is conserved along the fluid trajectories.*

The corresponding integral law is obtained as follows. Let c be a closed curve in the fluid, bounding a 2-surface S; and let c_τ be the curve obtained by moving each point of c a proper time τ along the fluid trajectory through that point. From the relation

$$\mathcal{L}_{\mathbf{u}}\omega_{\alpha\beta} = \nabla_\alpha\mathcal{L}_{\mathbf{u}}(h\,u_\beta) - \nabla_\beta\mathcal{L}_{\mathbf{u}}(h\,u_\alpha), \tag{1.51}$$

we have

$$\begin{aligned} 0 = \int_S \mathcal{L}_{\mathbf{u}}\omega_{\alpha\beta}dS^{\alpha\beta} &= \int_c \mathcal{L}_{\mathbf{u}}(h\,u_\alpha)dl^\alpha \\ &= \frac{d}{d\tau}\int_{c_\tau} h\,u_\alpha\,dl^\alpha, \end{aligned} \tag{1.52}$$

where Stokes's theorem was used to obtain the second equality and Eq. (A.83) of Appendix A was used in the last equality. That is, the line integral,

$$\int_{c_\tau} h\,u_\alpha\,dl^\alpha = \int_{c_\tau}\frac{\epsilon+p}{\rho}\,u_\alpha\,dl^\alpha \tag{1.53}$$

(the *circulation* of the flow along a closed curve), is independent of τ, conserved by the fluid flow.

Circular flow (absence of meridional circulation). Although newly born neutron stars are baroclinic, having meridional circulation and strong convection in the outer layers, as the star cools below the Fermi temperature for neutrons, its equation of state becomes essentially barotropic. The velocity field becomes circular (its only spatial velocity component is u^ϕ); viscosity and the magnetic field enforce uniform rotation.

1.2 The spacetime of a rotating star

A rotating star can be modeled by a stationary, axisymmetric, perfect-fluid spacetime, whose circular velocity field u^α can be written in terms of the two Killing vectors t^α and ϕ^α,

$$u^\alpha = u^t(t^\alpha + \Omega\phi^\alpha), \tag{1.54}$$

where the scalar

$$u^t := [-g_{\alpha\beta}(t^\alpha+\Omega\phi^\alpha)(t^\beta+\Omega\phi^\beta)]^{-1/2} \tag{1.55}$$