Percolation

Percolation theory was initiated some 50 years ago as a mathematical framework for the study of random physical processes such as flow through a disordered porous medium. It has proved to be a remarkably rich theory, with applications beyond natural phenomena to topics such as the theory of networks. Mathematically, it has many deep and difficult theorems, with a host of open problems remaining. The aims of this book are twofold. First, to present classical results, including the fundamental theorems of Harris, Kesten, Menshikov, Aizenman and Newman, in a way that is accessible to non-specialists. These results are presented with relatively simple proofs, making use of combinatorial techniques. Second, the authors describe, for the first time in a book, recent results of Smirnov on conformal invariance, and outline the proof that the critical probability for Voronoi percolation in the plane is 1/2.

Throughout, the presentation is streamlined, with elegant and straightforward proofs requiring minimal background in probability and graph theory, so that readers can quickly get up to speed. Numerous examples illustrate the important concepts and enrich the arguments. All in all, the book will be an essential purchase for mathematicians, probabilists, physicists, electrical engineers and computer scientists alike.

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To Gabriella and Gesine

Contents

Preface

Percolation theory was founded by Broadbent and Hammersley [1957] almost half a century ago; by now, thousands of papers and many books have been devoted to the subject. The original aim was to open up to mathematical analysis the study of random physical processes such as the flow of a fluid through a disordered porous medium. These *bona fide* problems in applied mathematics have attracted the attention of many physicists as well as pure mathematicians, and have led to the accumulation of much experimental and heuristic evidence for many remarkable phenomena. Mathematically, the subject has turned out to be much more difficult than might have been expected, with several deep results proved and many more conjectured.

The first spectacular mathematical result in percolation theory was proved by Kesten: in 1980 he complemented Harris's 1960 lower bound on the critical probability for bond percolation on the square lattice, and so proved that this critical probability is 1/2. To present this result, and numerous related results, Kesten [1982] published the first monograph devoted to the mathematical theory of percolation, concentrating on discrete two-dimensional percolation. A little later, Chayes and Chayes [1986b] came close to publishing the next book on the topic when they wrote an elegant and very long review article on percolation theory understood in a much broader sense.

For nearly two decades, Grimmett's 1989 book (with a second edition published in 1999) has been the standard reference for much of the basic theory of percolation on lattices. Other notable books on various aspects of percolation theory have been published by Smythe and Wierman [1978], Durrett [1988], Hughes [1995; 1996] and Meester and Roy [1996]; valuable survey articles have been written by Durrett [1984], Chayes, Puha and Sweet [1999], Kesten [2003] and Grimmett [2004], among others.

Our aims in this book are two-fold. First, we aim to present the 'classical' results of percolation in a way that is accessible to the non-specialist. To get straight to the point, we start with the best known result in the subject, the fundamental theorem of Harris and Kesten, even though this is a special case of later and more general results given in subsequent chapters.

The proof of the Harris–Kesten Theorem, in particular the upper bound due to Kesten, was a great achievement, and his proof not simple. Since then, however, especially with the advent of new tools in probabilistic combinatorics, many simple proofs have been found, a fact that most non-specialists are not aware of. For some of these arguments, all the pieces have been published some time ago, but perhaps not in one place, or only as comments that are easy to miss. Here, we bring together these various pieces and also more recent, very simple proofs of the Harris–Kesten Theorem.

In Chapters 4 and 5 we describe the very general results of Menshikov, and of Aizenman, Kesten and Newman; these results are again classical. Our aim here is to present them in the greatest generality that does not complicate the proofs.

Our second aim is to present recent results that have not yet appeared in book form. We give a complete proof of Smirnov's famous conformal invariance result; to our knowledge, no such account, with the i's dotted and t's crossed, has previously appeared. We finish by presenting an outline of the recent proof that the critical probability for random Voronoi percolation in the plane is 1/2.

As is often the case, we have tried to write the kind of book we should like to read. By now, percolation theory is an immensely rich subject with enough material for a dozen books, so it is not surprising that the choice of topics strongly reflects our tastes and interests. We have striven to give streamlined proofs, and to bring out the elegance of the arguments. To make the book accessible to as wide a readership as possible, we have assumed very little mathematical background and have illustrated the important concepts and main arguments with numerous examples.

In writing this book we have received help from many people. Paul Balister, Gesine Grosche, Svante Janson, Henry Liu, Robert Morris, Amites Sarkar, Alex Scott and Mark Walters were kind enough to read parts of the manuscript and to correct many misprints; for the many that remain, we apologize.