

Cambridge University Press

978-0-521-87228-7 - The Art of Mathematics: Coffee Time in Memphis

Bela Bollobas

Frontmatter

[More information](#)

## The Art of Mathematics

Can a Christian escape from a lion? How quickly can a rumour spread? Can you fool an airline into accepting oversize baggage? Recreational mathematics is full of frivolous questions in which the mathematician's art can be brought to bear. But play often has a purpose, whether it's bear cubs in mock fights, or war games. In mathematics, it can sharpen skills, or provide amusement, or simply surprise, and collections of problems have been the stock-in-trade of mathematicians for centuries. Two of the twentieth century's greatest players of problem posing and solving, Erdős and Littlewood, are the inspiration for this collection, which is designed to be sipped from, rather than consumed in one sitting. The questions themselves range in difficulty: the most challenging offer a glimpse of deep results that engage mathematicians today; even the easiest are capable of prompting readers to think about mathematics. All come with solutions, most with hints, and many with illustrations.

Whether you are an expert, or a beginner or an amateur, this book will delight for a lifetime.

BÉLA BOLLOBÁS is a Senior Research Fellow at Trinity College, Cambridge and holds the Jabie Hardin Chair of Excellence in Combinatorics at the University of Memphis. He has held visiting positions from Seattle to Singapore, from Brazil to Zurich. This is his tenth book.

GABRIELLA BOLLOBÁS is an eminent portraitist. She has sculpted busts of many outstanding scientists, including Dirac, von Neumann, Hardy, Littlewood, Erdős and Hawking. She lives in Cambridge with her husband.

Cambridge University Press

978-0-521-87228-7 - The Art of Mathematics: Coffee Time in Memphis

Bela Bollobas

Frontmatter

[More information](#)

*In re mathematica ars proponendi quaestionem pluris facienda est quam solvendi.*

Georg Cantor



*Coffee Time in Memphis*

Cambridge University Press

978-0-521-87228-7 - The Art of Mathematics: Coffee Time in Memphis

Bela Bollobas

Frontmatter

[More information](#)

The Art of Mathematics  
*Coffee Time in Memphis*

Béla Bollobás  
*University of Memphis and  
University of Cambridge*



Cambridge University Press

978-0-521-87228-7 - The Art of Mathematics: Coffee Time in Memphis

Bela Bollobas

Frontmatter

[More information](#)

CAMBRIDGE UNIVERSITY PRESS

Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press

The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

[www.cambridge.org](http://www.cambridge.org)

Information on this title: [www.cambridge.org/9780521872286](http://www.cambridge.org/9780521872286)

© Cambridge University Press 2006

This publication is in copyright. Subject to statutory exception and to the provisions of relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Cambridge University Press.

First published 2006

Reprinted with corrections 2007

Printed in the United Kingdom at the University Press, Cambridge

*A catalog record for this publication is available from the British Library*

ISBN 978-0-521-87228-7 hardback

ISBN 978-0-521-69395-0 paperback

Cambridge University Press has no responsibility for the persistence or accuracy of URLs for external or third-party internet websites referred to in this publication, and does not guarantee that any content on such websites is, or will remain, accurate or appropriate.

Cambridge University Press

978-0-521-87228-7 - The Art of Mathematics: Coffee Time in Memphis

Bela Bollobas

Frontmatter

[More information](#)

---

*To*

*J.E. Littlewood (1885–1977)*

*and*

*Paul Erdős (1913–1996)*

Cambridge University Press

978-0-521-87228-7 - The Art of Mathematics: Coffee Time in Memphis

Bela Bollobas

Frontmatter

[More information](#)

## Contents

	<i>Preface</i>	xiii
<b>1</b>	<b>The Problems</b>	1
<b>2</b>	<b>The Hints</b>	37
<b>3</b>	<b>The Solutions</b>	46
	1. The Lion and The Christian	46
	2. Integer Sequences: Erdős Problems for Epsilons	49
	3. Points on a Circle	51
	4. Partitions into Closed Sets	53
	5. Triangles and Squares	54
	6. Polygons and Rectangles	56
	7. African Rally	57
	8. Fixing Convex Domains	59
	9. Nested Subsets	62
	10. Almost Disjoint Subsets	64
	11. Loaded Dice	65
	12. An Unexpected Inequality	66
	13. Colouring Lines: the Erdős–Selfridge Theorem	67
	14. Independent Sets	69
	15. Expansion into Sums $2^i 3^j$	70
	16. A Tennis Match	71
	17. A Triangle Inequality: Another Erdős Problem for Epsilons	72
	18. Planar Domains of Diameter 1	74
	19. Orienting Graphs	75
	20. A Simple Clock	76
	21. Neighbours in a Matrix	77
	22. Separately Continuous Functions	78
	23. Boundary Cubes	79
	24. Lozenge Tilings	80

Cambridge University Press

978-0-521-87228-7 - The Art of Mathematics: Coffee Time in Memphis

Bela Bollobas

Frontmatter

[More information](#)

viii

*Contents*

25.	A Continuum Independent Set	84
26.	Separating Families of Sets	85
27.	Bipartite Covers of Complete Graphs	87
28.	Convexity and Intersecting Simplices: the Theorems of Radon and Carathéodory	89
29.	Intersecting Convex Sets: Helly's Theorem	91
30.	Judicious Partitions of Points	93
31.	Further Lozenge Tilings	94
32.	Two Squares in a Square	96
33.	Lines Through Points: the Sylvester–Gallai Theorem	99
34.	The Spread of Infection on a Square Grid	105
35.	The Spread of Infection in a $d$ -dimensional Box	107
36.	Sums of Integers: an Easy Erdős Problem for Epsilons	111
37.	Normal Numbers: the Champernowne Number	112
38.	Random Walks on Graphs	114
39.	Simple Tilings of Rectangles	115
40.	$L$ -tilings	117
41.	Antipodal Points and Maps: Borsuk's Theorem	118
42.	Bodies of Diameter 1: Borsuk's Problem	121
43.	Equilateral Triangles: Napoleon's Theorem	125
44.	Trisectors of Angles: Morley's Theorem	127
45.	Connected Subgraphs	130
46.	Subtrees of an Infinite Tree	134
47.	Two-distance Sets	135
48.	Gossiping Dons	137
49.	Exact Covers: the de Bruijn–Erdős Theorem	141
50.	Constant Intersections: an Extension of the de Bruijn–Erdős Theorem	143
51.	Bell Numbers	145
52.	Circles Touching a Square	148
53.	Gambling	150
54.	Complex Sequences	152
55.	Partitions of Integers	154
56.	Emptying Glasses	158
57.	Distances in Planar Sets	160
58.	Monic Polynomials	162
59.	Odd Clubs	164
60.	A Politically Correct Town	165
61.	Lattice Paths	166
62.	Triangulations of Polygons	169

Cambridge University Press

978-0-521-87228-7 - The Art of Mathematics: Coffee Time in Memphis

Bela Bollobas

Frontmatter

[More information](#)*Contents*

ix

63.	A Converse of Cauchy's Inequality: Zagier's Inequality	170
64.	Squares Touching a Square	171
65.	Infection with Three Neighbours	172
66.	The Spread of Infection on a Torus	174
67.	Dominating Sequences	175
68.	Sums of Reciprocals	176
69.	Absent-minded Passengers	177
70.	Airline Luggage	178
71.	Intersecting Sets: the Erdős–Ko–Rado Theorem	180
72.	Sperner Families: the MYBL Inequality	181
73.	Breaking a Stick	184
74.	Triads	185
75.	Colouring Complete Graphs	187
76.	Symmetric Convex Domains: a Theorem of Besicovitch	188
77.	Independent Random Variables	191
78.	Triangles Touching a Triangle	193
79.	Even and Odd Graphs	194
80.	Packing Squares: the Moon–Moser Theorem	195
81.	Filling a Matrix	198
82.	An Inequality Concerning Triangles: the Erdős–Mordell Theorem	200
83.	Perfect Difference Sets	204
84.	Difference Bases	206
85.	Satisfied Cricketers: the Hardy–Littlewood Maximal Theorem	209
86.	Random Words	213
87.	Crossing a Chess Board	215
88.	Powers of Paths and Cycles	217
89.	Powers of Oriented Cycles	218
90.	Perfect Trees	219
91.	Circular Sequences	221
92.	Infinite Sets with Integral Distances	223
93.	Finite Sets with Integral Distances	224
94.	Cube-free Words: Thue's Theorem	225
95.	Square-free Words: the Thue–Morse Theorem	227
96.	Addition of Residue Classes: the Cauchy–Davenport Theorem	230
97.	Sums Congruent to Zero: the Erdős–Ginzburg–Ziv Theorem	233
98.	Subwords of Distinct Words	238



Cambridge University Press

978-0-521-87228-7 - The Art of Mathematics: Coffee Time in Memphis

Bela Bollobas

Frontmatter

[More information](#)

x

*Contents*

99.	Prime Factors of Sums	239
100.	Catalan Numbers	241
101.	Permutations without Long Decreasing Subsequences	243
102.	Random Intervals: a Theorem of Justicz, Scheinerman and Winkler	245
103.	Sums of Convex Bodies: the Brunn–Minkowski Inequality	247
104.	Cross-Intersecting Families: Bollobás’s Lemma	249
105.	Saturated Hypergraphs	253
106.	The Norm of Averages: Hardy’s Inequality	254
107.	The Average of Geometric Means: Carleman’s Inequality	258
108.	Triangulating Squares	260
109.	Strongly Separating Families	263
110.	Strongly Separating Systems of Pairs of Sets	264
111.	The Maximum Edge-Boundary of a Down-set	266
112.	Partitioning a Subset of the Cube	268
113.	Weakly Cross-intersecting Pairs: Frankl’s Theorem	270
114.	Even Sets with Even Intersections	272
115.	Sets with Even Intersections	274
116.	Even Clubs	276
117.	Covering the Sphere	277
118.	The Kneser Graph: Lovász’s Theorem	278
119.	Partitions into Bricks	280
120.	Drawing Dense Graphs	281
121.	Unit Distances: Székely’s Theorem	283
122.	Point–Line Incidences	285
123.	Geometric Graphs without Parallel Edges	286
124.	Shortest Tours	289
125.	Density of Integers	292
126.	Black and White Sheep: Kirchberger’s Theorem	294
127.	Chords of Convex Bodies	295
128.	Neighbourly Polyhedra	297
129.	Neighbourly Simplices: Perles’ Theorem	300
130.	The Rank of a Matrix	302
131.	Modular Intersecting $k$ -uniform Set Systems: a Theorem of Frankl and Wilson	304
132.	Families without Orthogonal Vectors	307
133.	A Counterexample to Borsuk’s Conjecture: the Kahn–Kalai Theorem	309
134.	Periodic Sequences	312
135.	Periodic Words: the Fine–Wilf Theorem	314

Cambridge University Press

978-0-521-87228-7 - The Art of Mathematics: Coffee Time in Memphis

Bela Bollobas

Frontmatter

[More information](#)*Contents*

xi

136.	Points on a Hemisphere: Wendel's Theorem	316
137.	Planar and Spherical Triangles	319
138.	Hobnails: Hadžiivanov's Theorem	320
139.	A Probabilistic Inequality	322
140.	Cube Slicing	323
141.	Measures on $[0, 1]$ : the Hobby–Rice Theorem	325
142.	Cutting a Necklace	327
143.	The Norm of an Operator: the Riesz–Thorin Interpolation Theorem	329
144.	Uniform Covers	333
145.	Projections of Bodies	334
146.	BTBT: the Box Theorem of Bollobás and Thomason	336
147.	Intersecting Uniform Set Systems: the Ray–Chaudhuri– Wilson Inequality	338
148.	Intersecting Set Systems: the Frankl–Wilson Inequality	341
149.	Maps from $S^n$	343
150.	Closed Covers of $S^n$ : Hopf's Theorem	345
151.	Spherical Pairs	346
152.	Realizing Distances	347
153.	A Closed Cover of $S^2$	349
154.	A Friendly Party: the Friendship Theorem of Erdős, Rényi and Sós	350
155.	Polarities in Projective Planes	353
156.	Permutations of Vectors: Steinitz's Theorem	354
157.	An American Story	357
158.	Conway's Angel and Devil Game	359
159.	The Point–Line Game	367

Cambridge University Press

978-0-521-87228-7 - The Art of Mathematics: Coffee Time in Memphis

Bela Bollobas

Frontmatter

[More information](#)

## Preface

When I was putting together this collection of problems, I always asked myself whether the two giants of mathematics I had the good fortune to know well, Paul Erdős and J.E. Littlewood, would have found the question interesting. Would they have felt enticed to think about it? Could they have *not* thought about it, whether they wanted to or not? I think that many of the problems that ended up in this volume are indeed of the kind Erdős and Littlewood would have found difficult not to think about; since this collection contains many problems they considered or even posed, this assertion may not be as preposterous as it seems.

I was not yet ten when I fell in love with mathematical problems. Growing up in Hungary, this love got plenty of encouragement, and when at fourteen I got to know Paul Erdős, the greatest problem poser the world has ever seen, my fate was sealed. He treated me and other young people to a variety of beautiful and fascinating problems, solved and unsolved; many of the solved ones I heard from him in my teens appear in this volume.

The impetus for putting together this collection of problems came much later, in Memphis, where, for a few years now, some of the local and visiting mathematicians have had the habit of having lunch together, followed by coffee and a mathematical problem or two in my office. After a while, this meeting became quite an institution, with the tacit understanding that I would provide espresso and chocolate to please the palate, and a problem or two for the delectation of the mind. A *sine qua non* was that the problem should be enjoyable. The problems arising at these sessions form the core of this collection, so that for many years the working title of this book was *Coffee Time in Memphis*. It was only when I came to publishing it that my friend David Tranah, and his colleagues at CUP, insisted on changing the somewhat frivolous title I suggested to its present lofty incarnation, with CTM relegated to the subtitle.

It should be emphasized that this is a rather haphazard collection: in addition to the original ‘coffee time’ problems, and those from Erdős and Littlewood, there are many frivolous mathematical puzzles and a few problems from my teaching days in Cambridge, where I used to produce example sheets *For the enthusiast*. All of them are *mathematics with fun*: this is the main reason for publishing them in a volume. But just as a bear-cub can acquire life skills through play, so the reader can learn skills for a mathematical life simply by solving or just trying to solve the problems. The uninitiated can get a glimpse of how mathematical research is done and how the

Cambridge University Press

978-0-521-87228-7 - The Art of Mathematics: Coffee Time in Memphis

Bela Bollobas

Frontmatter

[More information](#)

mathematician's art is a combination of taste and technique; the professional can test his agility and ingenuity, 'the temper of his steel', as Hilbert said. Several of the problems are milestones in themselves or introduce the reader to serious areas of mathematics.

A few words of warning. This book is not a collection of *The Hundred Most Beautiful Theorems*, nor is it suitable for a systematic study of mathematics. Reading the book from cover to cover, as if it were an introduction to a branch of mathematics, may well cause indigestion and more damage than good.

Ideally, this collection should be used as a source of 'coffee time', enjoyable problems. The reader should pick out a problem or two to think about: if the problem is solved easily, fine, the next problem can come; but if it resists the initial attacks, the reader is likely to be even better off, for then the eventual solution (whether read or discovered) will be more pleasurable and beneficial. In particular, it is hoped that many of the questions can be used to inspire undergraduates taking standard, main-line, courses in mathematics.

The volume consists of three unequal parts. In the first part, over hundred and fifty problems are given, and the very brief second part contains hints to some of these problems. The third part is the longest by far: here the problems tend to be stated as theorems, and the solutions are given as proofs. Most solutions are followed by notes giving more information and references about the results. Although I have tried to give quite a few references, I have no doubt that some attributions should be more accurate and others are entirely missing: I apologize for these shortcomings. I shall be happy to correct any inaccuracies that brought to my attention.

Finally, it is a pleasure to acknowledge the help I have received in producing this book. First of all I should like to thank the 'regulars' and frequent visitors at coffee: Paul Balister, the mainstay of the round table, Stephen Kalikow, Vlado Nikiforov, Anthony Quas, Oliver Riordan, Amites Sarkar and Mark Walters, and the occasional visitors: Graham Brightwell, András Gyárfás, Ervin Győri, Imre Leader, Charles Read, Alex Scott, Miklós Simonovits, and many others. Without their love of mathematics and mathematical problems, this project would never even have been started. I should like to thank Paul Balister, József Balogh, Jonathan Cutler, Robert Morris, Vlado Nikiforov and Amites Sarkar for reading parts of the manuscript and helping me weed out a number of mistakes; for the many that no doubt remain, I apologize. For producing most of the figures, I am grateful to my invaluable and tireless assistant, Mrs. Tricia Simmons. Finally, I am most grateful to my wife, Gabriella, for putting the 'Art' into this book.

Béla Bollobás

*Cambridge, St. George's Day, 2006.*