The Art of Mathematics

Can a Christian escape from a lion? How quickly can a rumour spread? Can you fool an airline into accepting oversize baggage? Recreational mathematics is full of frivolous questions in which the mathematician's art can be brought to bear. But play often has a purpose, whether it's bear cubs in mock fights, or war games. In mathematics, it can sharpen skills, or provide amusement, or simply surprise, and collections of problems have been the stock-in-trade of mathematicians for centuries. Two of the twentieth century's greatest players of problem posing and solving, Erdős and Littlewood, are the inspiration for this collection, which is designed to be sipped from, rather than consumed in one sitting. The questions themselves range in difficulty: the most challenging offer a glimpse of deep results that engage mathematicians today; even the easiest are capable of prompting readers to think about mathematics. All come with solutions, most with hints, and many with illustrations.

Whether you are an expert, or a beginner or an amateur, this book will delight for a lifetime.

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GABRIELLA BOLLOBÁS is an eminent portraitist. She has sculpted busts of many outstanding scientists, including Dirac, von Neumann, Hardy, Littlewood, Erdős and Hawking. She lives in Cambridge with her husband.

In re mathematica ars proponendi quaestionem pluris facienda est quam solvendi.

Georg Cantor

Coffee Time in Memphis

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To

J.E. Littlewood (1885–1977)

and

Paul Erdős (1913–1996)

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Preface

When I was putting together this collection of problems, I always asked myself whether the two giants of mathematics I had the good fortune to know well, Paul Erdős and J.E. Littlewood, would have found the question interesting. Would they have felt enticed to think about it? Could they have *not* thought about it, whether they wanted to or not? I think that many of the problems that ended up in this volume are indeed of the kind Erdős and Littlewood would have found difficult not to think about; since this collection contains many problems they considered or even posed, this assertion may not be as preposterous as it seems.

I was not yet ten when I fell in love with mathematical problems. Growing up in Hungary, this love got plenty of encouragement, and when at fourteen I got to know Paul Erdős, the greatest problem poser the world has ever seen, my fate was sealed. He treated me and other young people to a variety of beautiful and fascinating problems, solved and unsolved; many of the solved ones I heard from him in my teens appear in this volume.

The impetus for putting together this collection of problems came much later, in Memphis, where, for a few years now, some of the local and visiting mathematicians have had the habit of having lunch together, followed by coffee and a mathematical problem or two in my office. After a while, this meeting became quite an institution, with the tacit understanding that I would provide espresso and chocolate to please the palate, and a problem or two for the delectation of the mind. A *sine qua non* was that the problem should be enjoyable. The problems arising at these sessions form the core of this collection, so that for many years the working title of this book was *Coffee Time in Memphis*. It was only when I came to publishing it that my friend David Tranah, and his colleagues at CUP, insisted on changing the somewhat frivolous title I suggested to its present lofty incarnation, with CTM relegated to the subtitle.

It should be emphasized that this is a rather haphazard collection: in addition to the original 'coffee time' problems, and those from Erdős and Littlewood, there are many frivolous mathematical puzzles and a few problems from my teaching days in Cambridge, where I used to produce example sheets *For the enthusiast.* All of them are *mathematics with fun*: this is the main reason for publishing them in a volume. But just as a bear-cub can acquire life skills through play, so the reader can learn skills for a mathematical life simply by solving or just trying to solve the problems. The unitiated can get a glimpse of how mathematical research is done and how the

mathematician's art is a combination of taste and technique; the professional can test his agility and ingenuity, 'the temper of his steel', as Hilbert said. Several of the problems are milestones in themselves or introduce the reader to serious areas of mathematics.

A few words of warning. This book is not a collection of *The Hundred Most Beautiful Theorems*, nor is it suitable for a systematic study of mathematics. Reading the book from cover to cover, as if it were an introduction to a branch of mathematics, may well cause indigestion and more damage than good.

Ideally, this collection should be used as a source of 'coffee time', enjoyable problems. The reader should pick out a problem or two to think about: if the problem is solved easily, fine, the next problem can come; but if it resists the initial attacks, the reader is likely to be even better off, for then the eventual solution (whether read or discovered) will be more pleasurable and beneficial. In particular, it is hoped that many of the questions can be used to inspire undergraduates taking standard, main-line, courses in mathematics.

The volume consists of three unequal parts. In the first part, over hundred and fifty problems are given, and the very brief second part contains hints to some of these problems. The third part is the longest by far: here the problems tend to be stated as theorems, and the solutions are given as proofs. Most solutions are followed by notes giving more information and references about the results. Although I have tried to give quite a few references, I have no doubt that some attributions should be more accurate and others are entirely missing: I apologize for these shortcomings. I shall be happy to correct any inaccuracies that brought to my attention.

Finally, it is a pleasure to acknowledge the help I have received in producing this book. First of all I should like to thank the 'regulars' and frequent visitors at coffee: Paul Balister, the mainstay of the round table, Stephen Kalikow, Vlado Nikiforov, Anthony Quas, Oliver Riordan, Amites Sarkar and Mark Walters, and the occasional visitors: Graham Brightwell, András Gyárfás, Ervin Győri, Imre Leader, Charles Read, Alex Scott, Miklós Simonovits, and many others. Without their love of mathematics and mathematical problems, this project would never even have been started. I should like to thank Paul Balister, József Balogh, Jonathan Cutler, Robert Morris, Vlado Nikiforov and Amites Sarkar for reading parts of the manuscript and helping me weed out a number of mistakes; for the many that no doubt remain, I apologize. For producing most of the figures, I am grateful to my invaluable and tireless assistant, Mrs. Tricia Simmons. Finally, I am most grateful to my wife, Gabriella, for putting the 'Art' into this book.

Béla Bollobás
Cambridge, St. George's Day, 2006.