Introduction to Algebraic Geometry

Algebraic geometry has a reputation for being difficult and inaccessible, even among mathematicians! This must be overcome. The subject is central to pure mathematics, and applications in fields like physics, computer science, statistics, engineering, and computational biology are increasingly important. This book is based on courses given at Rice University and the Chinese University of Hong Kong, introducing algebraic geometry to a diverse audience consisting of advanced undergraduate and beginning graduate students in mathematics, as well as researchers in related fields.

For readers with a grasp of linear algebra and elementary abstract algebra, the book covers the fundamental ideas and techniques of the subject and places these in a wider mathematical context. However, a full understanding of algebraic geometry requires a good knowledge of guiding classical examples, and this book offers numerous exercises fleshing out the theory. It introduces Gröbner bases early on and offers algorithms for almost every technique described. Both students of mathematics and researchers in related areas benefit from the emphasis on computational methods and concrete examples.

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To Eileen and William

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Preface

This book is an introduction to algebraic geometry, based on courses given at Rice University and the Institute of Mathematical Sciences of the Chinese University of Hong Kong from 2001 to 2006. The audience for these lectures was quite diverse, ranging from second-year undergraduate students to senior professors in fields like geometric modeling or differential geometry. Thus the algebraic prerequisites are kept to a minimum: a good working knowledge of linear algebra is crucial, along with some familiarity with basic concepts from abstract algebra. A semester of formal training in abstract algebra is more than enough, provided it touches on rings, ideals, and factorization. In practice, motivated students managed to learn the necessary algebra as they went along.

There are two overlapping and intertwining paths to understanding algebraic geometry. The first leads through sheaf theory, cohomology, derived functors and categories, and abstract commutative algebra – and these are just the prerequisites! We will not take this path. Rather, we will focus on specific examples and limit the formalism to what we need for these examples. Indeed, we will emphasize the strand of the formalism most useful for computations: We introduce *Gröbner bases* early on and develop algorithms for almost every technique we describe. The development of algebraic geometry since the mid 1990s vindicates this approach. The term 'Groebner' occurs in 1053 Math Reviews from 1995 to 2004, with most of these occurring in the last five years. The development of computers fast enough to do significant symbolic computations has had a profound influence on research in the field.

A word about what this book will *not* do: We develop computational techniques as a means to the end of learning algebraic geometry. However, we will not dwell on the technical questions of computability that might interest a computer scientist. We will also not spend time introducing the syntax of any particular computer algebra system. However, it is necessary that the reader be willing to carry out involved computations using elementary algebra, preferably with the help of a computer algebra system such as *Maple*, *Macaulay II*, or *Singular*.

Our broader goal is to display the core techniques of algebraic geometry in their natural habitat. These are developed systematically, with the necessary commutative algebra integrated with the geometry. Classical topics like resultants and elimination theory, are discussed in parallel with affine varieties, morphisms, and rational maps. Important examples of projective varieties (Grassmannians, Veronese varieties, Segre varieties) are emphasized, along with the matrix and exterior algebra needed to write down their defining equations.

It must be said that this book is not a comprehensive introduction to all of algebraic geometry. Shafarevich's book [37, 38] comes closest to this ideal; it addresses many important issues we leave untouched. Most other standard texts develop the material from a specific point of view, e.g., sheaf cohomology and schemes (Hartshorne [19]), classical geometry (Harris [17]), complex algebraic differential geometry (Griffiths and Harris [14]), or algebraic curves (Fulton [11]).

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The treatment of topics in this book owes a great deal to my teachers and the fine textbooks they have written: Serge Lang [27], Donal O'Shea [8], Joe Harris [17], David Eisenbud [9], and William Fulton [11]. My first exposure to algebraic geometry was through drafts of [8]; it has had a profound influence on how I teach the subject.