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Orbifolds and Stringy Topology

ALEJANDRO ADEM
University of British Columbia

JOHANN LEIDA
University of Wisconsin

YONGBIN RUAN
University of Michigan

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Introduction

Orbifolds lie at the intersection of many different areas of mathematics, including algebraic and differential geometry, topology, algebra, and string theory, among others. What is more, although the word "orbifold" was coined relatively recently,[1] orbifolds actually have a much longer history. In algebraic geometry, for instance, their study goes back at least to the Italian school under the guise of *varieties with quotient singularities*. Indeed, surface quotient singularities have been studied in algebraic geometry for more than a hundred years, and remain an interesting topic today. As with any other singular variety, an algebraic geometer aims to remove the singularities from an orbifold by either *deformation* or *resolution*. A deformation changes the defining equation of the singularities, whereas a resolution removes a singularity by blowing it up. Using combinations of these two techniques, one can associate many smooth varieties to a given singular one. In complex dimension two, there is a natural notion of a *minimal resolution*, but in general it is more difficult to understand the relationships between all the different desingularizations.

Orbifolds made an appearance in more recent advances towards Mori's birational geometric program in the 1980s. For Gorenstein singularities, the higher-dimensional analog of the minimal condition is the famous *crepant* resolution, which is minimal with respect to the canonical classes. A whole zoo of problems surrounds the relationship between crepant resolutions and Gorenstein orbifolds: this is often referred to as *McKay correspondence*. The McKay correspondence is an important motivation for this book; in complex dimension two it was solved by McKay himself. The higher-dimensional version has attracted increasing attention among algebraic geometers, and the existence of crepant resolutions in the dimension three case was eventually solved by an

[1] According to Thurston [148], it was the result of a democratic process in his seminar.

array of authors. Unfortunately, though, a Gorenstein orbifold of dimension four or more does not possess a crepant resolution in general. Perhaps the best-known example of a higher-dimensional crepant resolution is the Hilbert scheme of points of an algebraic surface, which forms a crepant resolution of its symmetric product. Understanding the cohomology of the Hilbert scheme of points has been an interesting problem in algebraic geometry for a considerable length of time.

Besides resolution, deformation also plays an important role in the classification of algebraic varieties. For instance, a famous conjecture of Reid [129] known as *Reid's fantasy* asserts that any two Calabi–Yau 3-folds are connected to each other by a sequence of resolutions or deformations. However, deformations are harder to study than resolutions. In fact, the relationship between the topology of a deformation of an orbifold and that of the orbifold itself is one of the major unresolved questions in orbifold theory.

The roots of orbifolds in algebraic geometry must also include the theory of *stacks*, which aims to deal with singular spaces by enlarging the concept of "space" rather than finding smooth desingularizations. The idea of an algebraic stack goes back to Deligne and Mumford [40] and Artin [7]. These early papers already show the need for the stack technology in fully understanding moduli problems, particularly the moduli stack of curves. Orbifolds are special cases of topological stacks, corresponding to "differentiable Deligne and Mumford stacks" in the terminology of [109].

Many of the orbifold cohomology theories we will study in this book have roots in and connections to cohomology theories for stacks. The book [90] of Laumon and Moret-Bailly is a good general reference for the latter. Orbifold Chen–Ruan cohomology, on the other hand, is closely connected to quantum cohomology – it is the *classical limit* of an orbifold quantum cohomology also due to Chen–Ruan. Of course, stacks also play an important role in the quantum cohomology of smooth spaces, since moduli stacks of maps from curves are of central importance in defining these invariants. For more on quantum cohomology, we refer the reader to McDuff and Salamon [107]; the original works of Kontsevich and Manin [87, 88], further developed in an algebraic context by Behrend [19] with Manin [21] and Fantechi [20], have also been very influential.

Stacks have begun to be studied in earnest by topologists and others outside of algebraic geometry, both in relation to orbifolds and in other areas. For instance, topological modular forms (tmf), a hot topic in homotopy theory, have a great deal to do with the moduli stack of elliptic curves [58].

Outside of algebraic geometry, orbifolds were first introduced into topology and differential geometry in the 1950s by Satake [138, 139], who called them *V-manifolds*. Satake described orbifolds as topological spaces generalizing smooth manifolds. In the same work, many concepts in smooth manifold theory such as de Rham cohomology, characteristic classes, and the Gauss–Bonnet theorem were generalized to V-manifolds. Although they are a useful concept for such problems as finite transformation groups, V-manifolds form a straightforward generalization of smooth manifolds, and can hardly be treated as a subject in their own right. This was reflected in the first twenty years of their existence. Perhaps the first inkling in the topological literature of additional features worthy of independent interest arose in Kawasaki's *V-manifold index theorem* [84, 85] where the index is expressed as a summation over the contribution of fixed point sets, instead of via a single integral as in the smooth case. This was the first appearance of the twisted sectors, about which we will have much more to say later.

In the late 1970s, V-manifolds were used seriously by Thurston in his geometrization program for 3-manifolds. In particular, Thurston invented the notion of an *orbifold fundamental group*, which was the first true invariant of an orbifold structure in the topological literature.[2] As noted above, it was during this period that the name V-manifold was replaced by the word orbifold. Important foundational work by Haefliger [64–68] and others inspired by foliation theory led to a reformulation of orbifolds using the language of groupoids. Of course, groupoids had also long played a central role in the development of the theory of stacks outlined above. Hence the rich techniques of groupoids can also be brought to bear on orbifold theory; in particular the work of Moerdijk [111–113] has been highly influential in developing this point of view. As a consequence of this, fundamental algebraic topological invariants such as classifying spaces, cohomology, bundles, and so forth have been developed for orbifolds.

Although orbifolds were already clearly important objects in mathematics, interest in them was dramatically increased by their role in string theory. In 1985, Dixon, Harvey, Vafa, and Witten built a conformal field theory model on singular spaces such as $\mathbb{T}^6/G$, the quotient of the six-dimensional torus by a smooth action of a finite group. In conformal field theory, one associates a Hilbert space and its operators to a manifold. For orbifolds, they made a surprising discovery: the Hilbert space constructed in the traditional fashion is not consistent, in the sense that its partition function is not modular. To recover modularity, they introduced additional Hilbert space factors to build a

[2] Of course, in algebraic geometry, invariants of orbifold structures (in the guise of stacks) appeared much earlier. For instance, Mumford's calculation of the Picard group of the moduli stack of elliptic curves [117] was published in 1965.

stringy Hilbert space. They called these factors *twisted sectors*, which intuitively represent the contribution of singularities. In this way, they were able to build a *smooth* stringy theory out of a singular space. Orbifold conformal field theory is very important in mathematics and is an impressive subject in its own right. In this book, however, our emphasis will rather be on topological and geometric information.

The main topological invariant obtained from orbifold conformal field theory is the *orbifold Euler number*. If an orbifold admits a crepant resolution, the string theory of the crepant resolution and the orbifold's string theory are thought to lie in the same family of string theories. Therefore, the orbifold Euler number should be the same as the ordinary Euler number of a crepant resolution. A successful effort to prove this statement was launched by Roan [131, 132], Batyrev and Dais [17], Reid [130] and others. In the process, the orbifold Euler number was extended to an orbifold Hodge number. Using intuition from physics, Zaslow [164] essentially discovered the correct stringy cohomology group for a global quotient using ad hoc methods. There was a very effective motivic integration program by Denef and Loeser [41, 42] and Batyrev [14, 16] (following ideas of Kontsevich [86]) that systematically established the equality of these numbers for crepant resolutions. On the other hand, motivic integration was not successful in dealing with finer structures, such as cohomology and its ring structure.

In this book we will focus on explaining how this problem was dealt with in the joint work of one of the authors (Ruan) with Chen [38]. Instead of guessing the correct formulation for the cohomology of a crepant resolution from orbifold data, Chen and Ruan approached the problem from the sigma-model quantum cohomology point of view, where the starting point is the space of maps from a Riemann surface to an orbifold. The heart of this approach is a correct theory of *orbifold morphisms*, together with a classification of those having domain an orbifold Riemann surface. The most surprising development is the appearance of a new object – the *inertia orbifold* – arising naturally as the target of an evaluation map, where for smooth manifolds one would simply recover the manifold itself. The key conceptual observation is that the components of the inertia orbifold should be considered the geometric realization of the conformal theoretic twisted sectors. This realization led to the successful construction of an orbifold quantum cohomology theory [37], and its classical limit leads to a new cohomology theory for orbifolds. The result has been a new wave of activity in the study of orbifolds. One of the main goals of this book is to give an account of *Chen–Ruan cohomology* which is accessible to students. In particular, a detailed treatment of orbifold morphisms is one of our basic themes.

Besides appearing in Chen–Ruan cohomology, the inertia orbifold has led to interesting developments in other orbifold theories. For instance, as first discussed in [5], the twisted sectors play a big part in orbifold K-theory and twisted orbifold K-theory. Twisted K-theory is a rapidly advancing field; there are now many types of twisting to consider, as well as interesting connections to physics [8, 54, 56].

We have formulated a basic framework that will allow a graduate student to grasp those essential aspects of the theory which play a role in the work described above. We have also made an effort to develop the background from a variety of viewpoints. In Chapter 1, we describe orbifolds very explicitly, using their manifold-like properties, their incarnations as groupoids, and, last but not least, their aspect as singular spaces in algebraic geometry. In Chapter 2, we develop the classical notions of cohomology, bundles, and morphisms for orbifolds using the techniques of Lie groupoid theory. In Chapter 3, we describe an approach to orbibundles and (twisted) K-theory using methods from equivariant algebraic topology. In Chapter 4, the heart of this book, we develop the Chen–Ruan cohomology theory using the technical background developed in the previous chapters. Finally, in Chapter 5 we describe some significant calculations for this cohomology theory.

As the theory of orbifolds involves mathematics from such diverse areas, we have made a selection of topics and viewpoints from a large and rather opaque menu of options. As a consequence, we have doubtless left out important work by many authors, for which we must blame our ignorance. Likewise, some technical points have been slightly tweaked to make the text more readable. We urge the reader to consult the original references.

It is a pleasure for us to thank the Department of Mathematics at the University of Wisconsin-Madison for its hospitality and wonderful working conditions over many years. All three of us have mixed feelings about saying farewell to such a marvelous place, but we must move on. We also thank the National Science Foundation for its support over the years. Last but not least, all three authors want to thank their wives for their patient support during the preparation of this manuscript. This text is dedicated to them.