1
Foundations

1.1 Classical effective orbifolds

Orbifolds are traditionally viewed as singular spaces that are locally modeled on a quotient of a smooth manifold by the action of a finite group. In algebraic geometry, they are often referred to as varieties with quotient singularities. This second point of view treats an orbifold singularity as an intrinsic structure of the space. For example, a codimension one orbifold singularity can be treated as smooth, since we can remove it by an analytic change of coordinates. This point of view is still important when we consider resolutions or deformations of orbifolds. However, when working in the topological realm, it is often more useful to treat the singularities as an additional structure – an *orbifold structure* – on an underlying space in the same way that we think of a smooth structure as an additional structure on a topological manifold. In particular, a topological space is allowed to have several different orbifold structures. Our introduction to orbifolds will reflect this latter viewpoint; the reader may also wish to consult the excellent introductions given by Moerdijk [112, 113].

The original definition of an orbifold was due to Satake [139], who called them V-*manifolds*. To start with, we will provide a definition of *effective* orbifolds equivalent to Satake's original one. Afterwards, we will provide a refinement which will encompass the more modern view of these objects. Namely, we will also seek to explain their definition using the language of groupoids, which, although it has the drawback of abstractness, does have important technical advantages. For one thing, it allows us to easily deal with ineffective orbifolds, which are generically singular. Such orbifolds are unavoidable in nature. For instance, the moduli stack of elliptic curves [117] (see Example 1.17) has a $\mathbb{Z}/2\mathbb{Z}$ singularity at a generic point. The definition below appears in [113].

Definition 1.1 Let X be a topological space, and fix $n \geq 0$.

- An n-dimensional *orbifold chart* on X is given by a connected open subset $\widetilde{U} \subseteq \mathbb{R}^n$, a finite group G of smooth automorphisms of $\widetilde{U}$, and a map $\phi : \widetilde{U} \to X$ so that ϕ is G-invariant and induces a homeomorphism of $\widetilde{U}/G$ onto an open subset $U \subseteq X$.
- An *embedding* $\lambda : (\widetilde{U}, G, \phi) \hookrightarrow (\widetilde{V}, H, \psi)$ between two such charts is a smooth embedding $\lambda : \widetilde{U} \hookrightarrow \widetilde{V}$ with $\psi\lambda = \phi$.
- An *orbifold atlas* on X is a family $\mathcal{U} = \{(\widetilde{U}, G, \phi)\}$ of such charts, which cover X and are locally compatible: given any two charts $(\widetilde{U}, G, \phi)$ for $U = \phi(\widetilde{U}) \subseteq X$ and $(\widetilde{V}, H, \psi)$ for $V \subseteq X$, and a point $x \in U \cap V$, there exists an open neighborhood $W \subseteq U \cap V$ of x and a chart $(\widetilde{W}, K, \mu)$ for W such that there are embeddings $(\widetilde{W}, K, \mu) \hookrightarrow (\widetilde{U}, G, \phi)$ and $(\widetilde{W}, K, \mu) \hookrightarrow (\widetilde{V}, H, \psi)$.
- An atlas $\mathcal{U}$ is said to *refine* another atlas $\mathcal{V}$ if for every chart in $\mathcal{U}$ there exists an embedding into some chart of $\mathcal{V}$. Two orbifold atlases are said to be *equivalent* if they have a common refinement.

We are now ready to provide a definition equivalent to the classical definition of an effective orbifold.

Definition 1.2 An *effective orbifold* $\mathcal{X}$ of dimension n is a paracompact Hausdorff space X equipped with an equivalence class $[\mathcal{U}]$ of n-dimensional orbifold atlases.

There are some important points to consider about this definition, which we now list. Throughout this section we will always assume that our orbifolds are effective.

1. We are assuming that for each chart $(\widetilde{U}, G, \phi)$, the group G is acting smoothly and effectively[1] on $\widetilde{U}$. In particular G will act freely on a dense open subset of $\widetilde{U}$.
2. Note that since smooth actions are locally smooth (see [31, p. 308]), any orbifold has an atlas consisting of linear charts, by which we mean charts of the form $(\mathbb{R}^n, G, \phi)$, where G acts on $\mathbb{R}^n$ via an orthogonal representation $G \subset O(n)$.
3. The following is an important technical result for the study of orbifolds (the proof appears in [113]): given two embeddings of orbifold charts $\lambda, \mu : (\widetilde{U}, G, \phi) \hookrightarrow (\widetilde{V}, H, \psi)$, there exists a unique $h \in H$ such that $\mu = h \cdot \lambda$.

[1] Recall that a group action is *effective* if $gx = x$ for all x implies that g is the identity. For basic results on topological and Lie group actions, we refer the reader to Bredon [31] and tom Dieck [152].

4. As a consequence of the above, an embedding of orbifold charts $\lambda : (\widetilde{U}, G, \phi) \hookrightarrow (\widetilde{V}, H, \psi)$ induces an injective group homomorphism, also denoted by $\lambda : G \hookrightarrow H$. Indeed, any $g \in G$ can be regarded as an embedding from $(\widetilde{U}, G, \phi)$ into itself. Hence for the two embeddings λ and $\lambda \cdot g$, there exists a unique $h \in H$ such that $\lambda \cdot g = h \cdot \lambda$. We denote this element $h = \lambda(g)$; clearly this correspondence defines the desired monomorphism.
5. Another key technical point is the following: given an embedding as above, if $h \in H$ is such that $\lambda(\widetilde{U}) \cap h \cdot \lambda(\widetilde{U}) \neq \emptyset$, then $h \in \operatorname{im} \lambda$, and so $\lambda(\widetilde{U}) = h \cdot \lambda(\widetilde{U})$.
6. If $(\widetilde{U}, G, \phi)$ and $(\widetilde{V}, H, \psi)$ are two charts for the same orbifold structure on X, and if $\widetilde{U}$ is simply connected, then there exists an embedding $(\widetilde{U}, G, \phi) \hookrightarrow (\widetilde{V}, H, \psi)$ whenever $\phi(\widetilde{U}) \subset \psi(\widetilde{V})$.
7. Every orbifold atlas for X is contained in a unique maximal one, and two orbifold atlases are equivalent if and only if they are contained in the same maximal one. As with manifolds, we tend to work with a maximal atlas.
8. If the finite group actions on all the charts are free, then X is locally Euclidean, hence a manifold.

Next we define the notion of smooth maps between orbifolds.

Definition 1.3 Let $\mathcal{X} = (X, \mathcal{U})$ and $\mathcal{Y} = (Y, \mathcal{V})$ be orbifolds. A map $f : X \to Y$ is said to be *smooth* if for any point $x \in X$ there are charts $(\widetilde{U}, G, \phi)$ around x and $(\widetilde{V}, H, \psi)$ around $f(x)$, with the property that f maps $U = \phi(\widetilde{U})$ into $V = \psi(\widetilde{V})$ and can be lifted to a smooth map $\tilde{f} : \widetilde{U} \to \widetilde{V}$ with $\psi \tilde{f} = f \phi$.

Using this we can define the notion of *diffeomorphism* of orbifolds.

Definition 1.4 Two orbifolds $\mathcal{X}$ and $\mathcal{Y}$ are *diffeomorphic* if there are smooth maps of orbifolds $f : X \to Y$ and $g : Y \to X$ with $f \circ g = 1_Y$ and $g \circ f = 1_X$.

A more stringent definition for maps between orbifolds is required if we wish to preserve fiber bundles (as well as sheaf-theoretic constructions) on orbifolds. The notion of an *orbifold morphism* will be introduced when we discuss orbibundles; for now we just wish to mention that a diffeomorphism of orbifolds is in fact an orbifold morphism, a fact that ensures that orbifold equivalence behaves as expected.

Let X denote the underlying space of an orbifold $\mathcal{X}$, and let $x \in X$. If $(\widetilde{U}, G, \phi)$ is a chart such that $x = \phi(y) \in \phi(\widetilde{U})$, let $G_y \subseteq G$ denote the *isotropy subgroup* for the point y. We claim that up to conjugation, this group does not depend on the choice of chart. Indeed, if we used a different chart, say $(\widetilde{V}, H, \psi)$, then by our definition we can find a third chart $(\widetilde{W}, K, \mu)$ around x together with embeddings $\lambda_1 : (\widetilde{W}, K, \mu) \hookrightarrow (\widetilde{U}, G, \phi)$ and $\lambda_2 : (\widetilde{W}, K, \mu) \hookrightarrow (\widetilde{V}, H, \psi)$. As we have seen, these inclusions are equivariant with respect to the induced injective group homomorphisms; hence the embeddings induce inclusions $K_y \hookrightarrow G_y$ and $K_y \hookrightarrow H_y$. Now applying property 5 discussed above, we see that these maps must also be onto, hence we have an isomorphism $H_y \cong G_y$. Note that if we chose a different preimage y', then G_y is conjugate to $G_{y'}$. Based on this, we can introduce the notion of a *local group* at a point $x \in X$.

Definition 1.5 Let $x \in X$, where $\mathcal{X} = (X, \mathcal{U})$ is an orbifold. If $(\widetilde{U}, G, \psi)$ is any local chart around $x = \psi(y)$, we define the *local group* at x as

$$G_x = \{g \in G \mid gy = y\}.$$

This group is uniquely determined up to conjugacy in G.

We now use the notion of local group to define the singular set of the orbifold.

Definition 1.6 For an orbifold $\mathcal{X} = (X, \mathcal{U})$, we define its *singular set* as

$$\Sigma(\mathcal{X}) = \{x \in X \mid G_x \neq 1\}.$$

This subspace will play an important role in what follows.

Before discussing any further general facts about orbifolds, it seems useful to discuss the most natural source of examples for orbifolds, namely, compact transformation groups. Let G denote a compact Lie group acting smoothly, effectively and *almost freely* (i.e., with finite stabilizers) on a smooth manifold M. Again using the fact that smooth actions on manifolds are locally smooth, we see that given $x \in M$ with isotropy subgroup G_x, there exists a chart $U \cong \mathbb{R}^n$ containing x that is G_x-invariant. The orbifold charts are then simply (U, G_x, π), where $\pi : U \to U/G_x$ is the projection map. Note that the quotient space $X = M/G$ is automatically paracompact and Hausdorff. We give this important situation a name.

Definition 1.7 An *effective quotient orbifold* $\mathcal{X} = (X, \mathcal{U})$ is an orbifold given as the quotient of a smooth, effective, almost free action of a compact Lie group G on a smooth manifold M; here $X = M/G$ and $\mathcal{U}$ is constructed from a manifold atlas using the locally smooth structure.

An especially attractive situation arises when the group G is finite; following established tradition, we single out this state of affairs.

Definition 1.8 If a finite group G acts smoothly and effectively on a smooth manifold M, the associated orbifold $\mathcal{X} = (M/G, \mathcal{U})$ is called an *effective global quotient*.

More generally, if we have a compact Lie group acting smoothly and almost freely on a manifold M, then there is a group extension

$$1 \to G_0 \to G \to G_{\text{eff}} \to 1,$$

where $G_0 \subset G$ is a finite group and G_{eff} acts *effectively* on M. Although the orbit spaces M/G and M/G_{eff} are identical, the reader should note that the structure on $X = M/G$ associated to the full G action will not be a classical orbifold, as the constant kernel G_0 will appear in all the charts. However, the main properties associated to orbifolds easily apply to this situation, an indication that perhaps a more flexible notion of orbifold is required – we will return to this question in Section 1.4. For a concrete example of this phenomenon, see Example 1.17.

1.2 Examples

Orbifolds are of interest from several different points of view, including representation theory, algebraic geometry, physics, and topology. One reason for this is the existence of many interesting examples constructed from different fields of mathematics. Many new phenomena (and subsequent new theorems) were first observed in these key examples, and they are at the heart of this subject.

Given a finite group G acting smoothly on a compact manifold M, the quotient M/G is perhaps the most natural example of an orbifold. We will list a number of examples which are significant in the literature, all of which arise as global quotients of an n-torus. To put them in context, we first describe a general procedure for constructing group actions on $\mathbb{T}^n = (\mathbb{S}^1)^n$. The group $GL_n(\mathbb{Z})$ acts by matrix multiplication on $\mathbb{R}^n$, taking the lattice $\mathbb{Z}^n$ to itself. This then induces an action on $\mathbb{T}^n = (\mathbb{R}/\mathbb{Z})^n$. In fact, one can easily show that the map induced by looking at the action in homology, $\Phi : \text{Aut}(\mathbb{T}^n) \to GL_n(\mathbb{Z})$, is a split surjection. In particular, if $G \subset GL_n(\mathbb{Z})$ is a finite subgroup, then this defines an effective G-action on $\mathbb{T}^n$. Note that by construction the G-action lifts to a proper action of a discrete group Γ on $\mathbb{R}^n$; this is an example of a *crystallographic group*, and it is easy to see that it fits into a group extension of the form $1 \to (\mathbb{Z})^n \to \Gamma \to G \to 1$. The first three examples are all special cases of this construction, but are worthy of special attention due to their role in geometry and physics (we refer the reader to [4] for a detailed discussion of this class of examples).

Example 1.9 Let $\mathcal{X} = \mathbb{T}^4/(\mathbb{Z}/2\mathbb{Z})$, where the action is generated by the involution τ defined by

$$\tau(e^{it_1}, e^{it_2}, e^{it_3}, e^{it_4}) = (e^{-it_1}, e^{-it_2}, e^{-it_3}, e^{-it_4}).$$

Note that under the construction above, τ corresponds to the matrix $-I$. This orbifold is called the *Kummer surface*, and it has sixteen isolated singular points.

Example 1.10 Let $\mathbb{T}^6 = \mathbb{C}^3/\Gamma$, where Γ is the lattice of integral points. Consider $(\mathbb{Z}/2\mathbb{Z})^2$ acting on $\mathbb{T}^6$ via a lifted action on $\mathbb{C}^3$, where the generators σ_1 and σ_2 act as follows:

$$\sigma_1(z_1, z_2, z_3) = (-z_1, -z_2, z_3),$$
$$\sigma_2(z_1, z_2, z_3) = (-z_1, z_2, -z_3),$$
$$\sigma_1\sigma_2(z_1, z_2, z_3) = (z_1, -z_2, -z_3).$$

Our example is $\mathcal{X} = \mathbb{T}^6/(\mathbb{Z}/2\mathbb{Z})^2$. This example was considered by Vafa and Witten [155].

Example 1.11 Let $\mathcal{X} = \mathbb{T}^6/(\mathbb{Z}/4\mathbb{Z})$. Here, the generator κ of $\mathbb{Z}/4\mathbb{Z}$ acts on $\mathbb{T}^6$ by

$$\kappa(z_1, z_2, z_3) = (-z_1, iz_2, iz_3).$$

This example has been studied by Joyce in [75], where he constructed five different desingularizations of this singular space. The importance of this accomplishment lies in its relation to a conjecture of Vafa and Witten, which we discuss in Chapter 4.

Algebraic geometry is another important source of examples of orbifolds. Our first example of this type is the celebrated *mirror quintic*.

Example 1.12 Suppose that Y is a degree five hypersurface of $\mathbb{C}P^4$ given by a homogeneous equation

$$z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 + \phi z_0 z_1 z_2 z_3 z_4 = 0, \tag{1.1}$$

where ϕ is a generic constant. Then Y admits an action of $(\mathbb{Z}/5\mathbb{Z})^3$. Indeed, let λ be a primitive fifth root of unity, and let the generators e_1, e_2, and e_3 of $(\mathbb{Z}/5\mathbb{Z})^3$ act as follows:

$$e_1(z_0, z_1, z_2, z_3, z_4) = (\lambda z_0, z_1, z_2, z_3, \lambda^{-1} z_4),$$
$$e_2(z_0, z_1, z_2, z_3, z_4) = (z_0, \lambda z_1, z_2, z_3, \lambda^{-1} z_4),$$
$$e_3(z_0, z_1, z_2, z_3, z_4) = (z_0, z_1, \lambda z_2, z_3, \lambda^{-1} z_4).$$

The quotient $\mathcal{X} = Y/(\mathbb{Z}/5\mathbb{Z})^3$ is called the *mirror quintic*.

Example 1.13 Suppose that M is a smooth manifold. One can form the *symmetric product* $X_n = M^n/S_n$, where the symmetric group S_n acts on M^n by permuting coordinates. Tuples of points have isotropy according to how many repetitions they contain, with the diagonal being the fixed point set. This set of examples has attracted a lot of attention, especially in algebraic geometry. For the special case when M is an algebraic surface, X_n admits a beautiful resolution, namely the Hilbert scheme of points of length n, denoted $X^{[n]}$. We will revisit this example later, particularly in Chapter 5.

Example 1.14 Let G be a finite subgroup of $GL_n(\mathbb{C})$ and let $\mathcal{X} = \mathbb{C}^n/G$; this is a singular complex manifold called a *quotient singularity*. $\mathcal{X}$ has the structure of an algebraic variety, arising from the algebra of G-invariant polynomials on $\mathbb{C}^n$. These examples occupy an important place in algebraic geometry related to McKay correspondence. In later applications, it will often be important to assume that $G \subset SL_n(\mathbb{C})$, in which case $\mathbb{C}^n/G$ is said to be *Gorenstein*. We note in passing that the Gorenstein condition is essentially the local version of the definition of *SL-orbifolds* given on page 15.

Example 1.15 Consider

$$\mathbb{S}^{2n+1} = \left\{ (z_0, \ldots, z_n) \mid \sum_i |z_i|^2 = 1 \right\} \subseteq \mathbb{C}^{n+1},$$

then we can let $\lambda \in \mathbb{S}^1$ act on it by

$$\lambda(z_0, \ldots, z_n) = (\lambda^{a_0} z_0, \ldots, \lambda^{a_n} z_n),$$

where the a_i are coprime integers. The quotient

$$\mathbb{WP}(a_0, \ldots, a_n) = \mathbb{S}^{2n+1}/\mathbb{S}^1$$

is called a *weighted projective space*, and it plays the role of the usual projective space in orbifold theory. $\mathbb{WP}(1, a)$, is the famous *teardrop*, which is the easiest example of a non-global quotient orbifold. We will use the orbifold fundamental group to establish this later.

Example 1.16 Generalizing from the teardrop to other two-dimensional orbifolds leads us to consider *orbifold Riemann surfaces*, a fundamental class of examples that are not hard to describe. We need only specify the (isolated) singular points and the order of the local group at each one. If x_i is a singular point with order m_i, it is understood that the local chart at x_i is $D/\mathbb{Z}_{m_i}$ where D is a small disk around zero and the action is $e \circ z = \lambda z$ for e the generator of $\mathbb{Z}_{m_i}$ and $\lambda^{m_i} = 1$.

Suppose that an orbifold Riemann surface Σ has genus g and k singular points. Thurston [149] has shown that it is a global quotient if and only if $g + 2k \geq 3$ or $g = 0$ and $k = 2$ with $m_1 = m_2$. In any case, an orbifold Riemann surface is always a quotient orbifold, as it can be expressed as $X^3/\mathbb{S}^1$, where X^3 is a 3-manifold called a *Seifert fiber manifold* (see [140] for more on Seifert manifolds).

Example 1.17 Besides considering orbifold structures on a single surface, we can also consider various moduli spaces – or rather, moduli *stacks* – of (non-orbifold) curves. As we noted in the introduction to this chapter, these were among the first orbifolds in which the importance of the additional structure (such as isotropy groups) became evident [7]. For simplicity, we describe the orbifold structure on the moduli space of elliptic curves.

For our purposes, elliptic curves may be defined to be those tori $\mathbb{C}/L$ obtained as the quotient of the complex numbers $\mathbb{C}$ by a lattice of the form $L = \mathbb{Z} + \mathbb{Z}\tau \subset \mathbb{C}^*$, where $\tau \in \mathbb{C}^*$ satisfies $\operatorname{im}\tau > 0$. Suppose we have two elliptic curves $\mathbb{C}/L$ and $\mathbb{C}/L'$, specified by elements τ and τ' in the Poincaré upper half plane $H = \{z \in \mathbb{C} \mid \operatorname{im} z > 0\}$. Then $\mathbb{C}/L$ and $\mathbb{C}/L'$ are isomorphic if there is a matrix in $SL_2(\mathbb{Z})$ that takes τ to τ', where the action is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}.$$

The moduli stack or orbifold of elliptic curves is then the quotient $H/SL_2(\mathbb{Z})$. This is a two-dimensional orbifold, although since the matrix $-\mathrm{Id}$ fixes every point of H, the action is not effective. We could, however, replace $G = SL_2(\mathbb{Z})$ by $G_{\text{eff}} = PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/(\pm\mathrm{Id})$ to obtain an associated effective orbifold. The only points with additional isotropy are the two points corresponding to $\tau = i$ and $\tau = e^{2\pi i/3}$ (which correspond to the square and hexagonal lattices, respectively). The first is fixed by a cyclic subgroup of $SL_2(\mathbb{Z})$ having order 4, while the second is fixed by one of order 6.

In Chapter 4, we will see that understanding certain moduli stacks involving orbifold Riemann surfaces is central to Chen–Ruan cohomology.

Example 1.18 Suppose that (Z, ω) is a symplectic manifold admitting a *Hamiltonian* action of the torus $\mathbb{T}^k$. This means that the torus is acting effectively by symplectomorphisms, and that there is a *moment map* $\mu : Z \to \mathfrak{t}^*$, where $\mathfrak{t}^* \cong \mathbb{R}^k$ is the dual of the Lie algebra $\mathfrak{t}$ of $\mathbb{T}^k$. Any $v \in \mathfrak{t}$ generates a one-parameter subgroup. Differentiating the action of this one-parameter subgroup, one obtains a vector field X_v on Z. The moment map is then related to the action by requiring the equation

$$\omega(X_v, X) = d\mu(X)(v)$$

to hold for each $X \in TZ$.

One would like to study $Z/\mathbb{T}^k$ as a symplectic space, but of course even if the quotient space is smooth, it will often fail to be symplectic: for instance, it could have odd dimension. To remedy this, take a regular value $c \in \mathbb{R}^k$ of μ. Then $\mu^{-1}(c)$ is a smooth submanifold of Z, and one can show that $\mathbb{T}^k$ acts on it. The quotient $\mu^{-1}(c)/\mathbb{T}^k$ will always have a symplectic structure, although it is usually only an orbifold and not a manifold. This orbifold is called the *symplectic reduction* or *symplectic quotient* of Z, and is denoted by $Z//\mathbb{T}^k$.

The symplectic quotient depends on the choice of the regular value c. If we vary c, there is a chamber structure for $Z//\mathbb{T}^k$ in the following sense. Namely, we can divide $\mathbb{R}^k$ into subsets called *chambers* so that inside each chamber, $Z//\mathbb{T}^k$ remains the same. When we cross a wall separating two chambers, $Z//\mathbb{T}^k$ will undergo a surgery operation similar to a flip in algebraic geometry. The relation between the topology of Z and that of $Z//\mathbb{T}^k$ and the relation between symplectic quotients in different chambers have long been interesting problems in symplectic geometry – see [62] for more information.

The construction of the symplectic quotient has an analog in algebraic geometry called the *geometric invariant theory (GIT) quotient*. Instead of $\mathbb{T}^k$, one has the complex torus $(\mathbb{C}^*)^k$. The existence of an action by $(\mathbb{C}^*)^k$ is equivalent to the condition that the induced action of $\mathbb{T}^k$ be Hamiltonian. The choice of c corresponds to the choice of an ample line bundle L such that the action of $(\mathbb{C}^*)^k$ lifts to the total space of L. Taking the level set $\mu^{-1}(c)$ corresponds to the choice of semi-stable orbits.

Example 1.19 The above construction can be used to construct explicit examples. A convenient class of examples are *toric varieties*, where $Z = \mathbb{C}^r$. The combinatorial datum used to define a Hamiltonian toric action is called a *fan*. Most explicit examples arising in algebraic geometry are complete intersections of toric varieties.

Example 1.20 Let G denote a Lie group with only finitely many components. Then G has a maximal compact subgroup K, unique up to conjugacy, and the homogeneous space $X = G/K$ is diffeomorphic to $\mathbb{R}^d$, where $d = \dim G - \dim K$. Now let $\Gamma \subset G$ denote a discrete subgroup. Γ has a natural left action on this homogeneous space; moreover, it is easy to check that this is a proper action, due to the compactness of K. Consequently, all the stabilizers $\Gamma_x \subseteq \Gamma$ are finite, and each $x \in X$ has a neighborhood U such that $\gamma U \cap U = \emptyset$ for $\gamma \in \Gamma \setminus \Gamma_x$. Clearly, this defines an orbifold structure on the quotient space X/Γ. We will call this type of example an *arithmetic orbifold*; they are of fundamental interest in many areas of mathematics, including topology and number theory. Perhaps the favorite example is the orbifold associated to $SL_n(\mathbb{Z})$, where the associated symmetric space on which it acts is $SL_n(\mathbb{R})/SO_n \cong \mathbb{R}^d$, with $d = \frac{1}{2}n(n-1)$.

1.3 Comparing orbifolds to manifolds

One of the reasons for the interest in orbifolds is that they have geometric properties akin to those of manifolds. A central topic in orbifold theory has been to elucidate the appropriate adaptations of results from manifold theory to situations involving finite group quotient singularities.

Given an orbifold $\mathcal{X} = (X, \mathcal{U})$ let us first consider how the charts are glued together to yield the space X. Given $(\widetilde{U}, G, \phi)$ and $(\widetilde{V}, H, \psi)$ with $x \in U \cap V$, there is by definition a third chart $(\widetilde{W}, K, \mu)$ and embeddings λ_1, λ_2 from this chart into the other two. Here W is an open set with $x \in W \subset U \cap V$. These embeddings give rise to diffeomorphisms $\lambda_1^{-1} : \lambda_1(\widetilde{W}) \to \widetilde{W}$ and $\lambda_2 : \widetilde{W} \to \lambda_2(\widetilde{W})$, which can be composed to provide an equivariant diffeomorphism $\lambda_2\lambda_1^{-1} : \lambda_1(\widetilde{W}) \to \lambda_2(\widetilde{W})$ between an open set in $\widetilde{U}$ and an open set in $\widetilde{V}$. The word "equivariant" needs some explanation: we are using the fact that an embedding is an equivariant map with respect to its associated injective group homomorphism, and that the local group K associated to $\widetilde{W}$ is isomorphic to the local groups associated to its images. Hence we can regard $\lambda_2\lambda_1^{-1}$ as an equivariant diffeomorphism of K-spaces. We can then proceed to glue $\widetilde{U}/G$ and $\widetilde{V}/H$ according to the induced homeomorphism of subsets, i.e., identify $\phi(\tilde{u}) \sim \psi(\tilde{v})$ if $\lambda_2\lambda_1^{-1}(\tilde{u}) = \tilde{v}$. Now let

$$Y = \bigsqcup_{\widetilde{U} \in \mathcal{U}} (\widetilde{U}/G)/\sim$$

be the space obtained by performing these identifications on the orbifold atlas. The maps $\phi : \widetilde{U} \to X$ induce a homeomorphism $\Phi : Y \to X$.

This procedure is, of course, an analog of what takes place for manifolds, except that our gluing maps are slightly more subtle. It is worth noting that we can think of $\lambda_2\lambda_1^{-1}$ as a *transition function*. Given another λ_1' and λ_2', we have seen that there must exist unique $g \in G$ and $h \in H$ such that $\lambda_1' = g\lambda_1$ and $\lambda_2' = h\lambda_2$. Hence the resulting transition function is $h\lambda_2\lambda_1^{-1}g^{-1}$. This can be restated as follows: there is a transitive $G \times H$ action on the set of all of these transition functions.

We now use this explicit approach to construct a *tangent bundle* for an orbifold $\mathcal{X}$. Given a chart $(\widetilde{U}, G, \phi)$, we can consider the tangent bundle $T\widetilde{U}$; note that by assumption G acts smoothly on $\widetilde{U}$, hence it will also act smoothly