CHAPTER 1

More Inflation Tilings

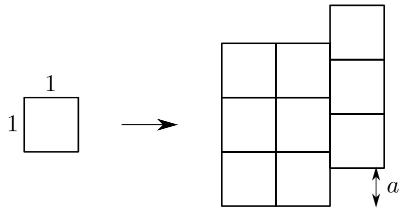
Dirk Frettlöh

Inflation tilings exhibit a wealth of properties, as we shall demonstrate by means of explicit examples. In this sense, this chapter can be seen as an extension of [AO1, Ch. 6]. Along the journey, the concept of inflation will be generalised in several ways. One of the aims of our exposition is to highlight some of the more exotic behaviour that can be observed in the realm of inflation tilings and to point out some interesting questions raised by these examples. Most of the examples discussed below are contained in the Tilings Encyclopedia [21].

1.1. A simple inflation tiling without FLC

Many if not most examples of aperiodic tilings in the literature have *finite local complexity* (FLC); see [11] or [AO1, Sec. 5.3] for background. Even though these examples may be easier to construct, there is no reason to assume that FLC is a typical property of inflation tilings. One of the simplest inflation tilings that fails to have FLC is generated by the following rule [33]:

(1.1.1)



The inflation factor (or multiplier) for this rule is 3, and the single prototile is a unit square. Under the inflation, each square is replaced by three columns of three squares each, where the third column is shifted vertically by some irrational number $a \notin \mathbb{Q}$. The resulting tilings contain pairs of squares sharing an entire edge, as well as pairs of squares sharing part of an edge, where vertical shifts of the form $na \bmod 1$ between adjacent squares are realised with infinitely many different $n \in \mathbb{N}$. In particular, the integer n takes the

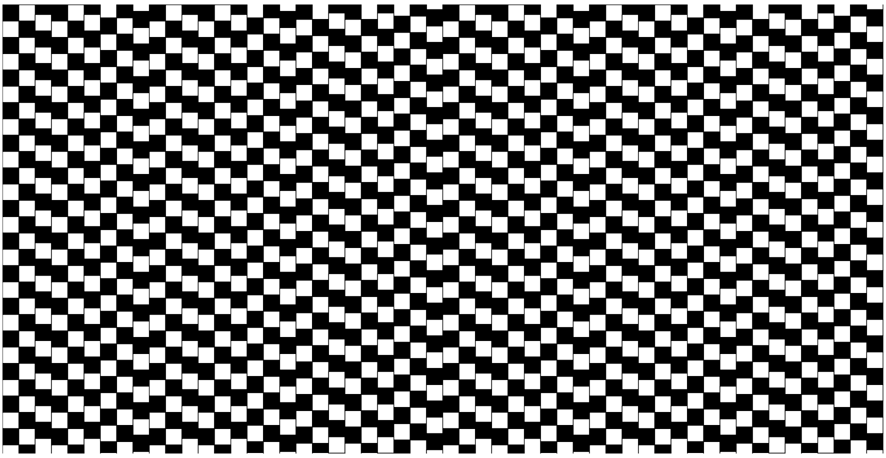
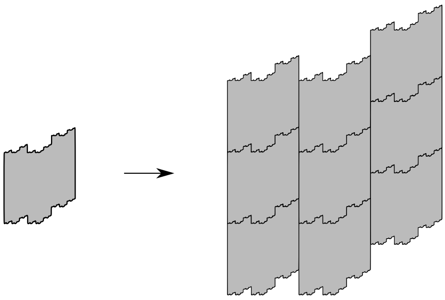


FIGURE 1.1.1. A patch of a simple non-FLC tiling, as defined by Eq. (1.1.1). For clarity, the square tiles are alternatingly coloured black and white.

values 1, $3+1$, 3^2+3+1 etc. Since a is irrational, the corresponding values of na mod 1 are all different. Consequently, there are infinitely many pairwise non-congruent clusters (or patches) of two adjacent tiles. This shows that the tilings obtained from this inflation rule do not have the FLC property. A patch of such a tiling is shown in Figure 1.1.1.

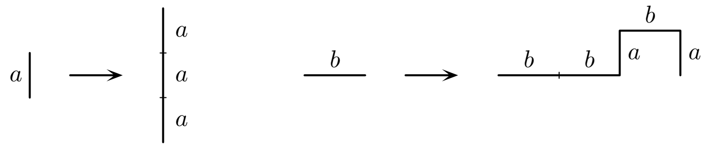
If one does not insist that the tiles are polygons, one can turn the inflation rule (1.1.1) into a *stone inflation* [AO1, p. 148]. Parts of the boundary of the prototile will then be turned into fractals. The corresponding stone inflation is given by



and is clearly mutually locally derivable (MLD) with the inflation (1.1.1); see [AO1, Sec. 5.2] for background on MLD as an equivalence relation.

The boundary of the prototile is not a 'proper' fractal, in the sense that its Hausdorff dimension is 1. This can be seen by employing the methods described in [47]. Denote the upper part of the boundary of the prototile by

F. The stone inflation induces a substitution σ for F, namely



where b denotes a horizontal line segment of unit length and a denotes a vertical line segment of length a. The matrix of this induced substitution is $M = \left(\begin{smallmatrix} 3 & 2 \\ 0 & 3 \end{smallmatrix}\right)$. The contraction factor of the iterated function system (IFS, compare [AO1, Sec. 7.1]) for F is $c = \frac{1}{3}$, the Perron–Frobenius (PF) eigenvalue of M is 3. Loosely speaking, the 'growth rate' of the iterates of the IFS is $3 \cdot \frac{1}{3} = 1$. In order to get a set with dimension strictly larger than 1, the growth rate needs to be > 1. For instance, the growth rate of the Koch curve is $\frac{4}{3}$, which results in its Hausdorff dimension being $\frac{\log(4)}{\log(3)}$.

More precisely, the curve F is only a subset of the solution S of the IFS corresponding to σ, because the IFS has overlaps, and these overlaps yield additional parts of S that are not part of the boundary of the prototile of the tiling. Nevertheless, F is a subset of S, hence its dimension is equal to or less than the dimension of S. Because F has at least dimension 1, it suffices to show that the Hausdorff dimension of S is 1, too. By [47, Prop. 6.106], the *affinity dimension* of S is

$$\dim_{\mathsf{aff}}(S) = \frac{\log(\rho(M))}{\log(c^{-1})} = \frac{\log(3)}{\log(3)} = 1.$$

Here, $\rho(M)$ denotes the spectral radius of M. Due to [47, Prop. 4.122], the Hausdorff dimension of S is bounded by the affinity dimension, hence it also equals 1. Consequently, the boundary curve F of the prototile has Hausdorff dimension 1 as well.

There are more sophisticated but essentially similar constructions of non-FLC inflation tilings by Danzer [6, 7] as well as by Frank and Robinson [14]; see also [AO1, Ex. 5.8] as well as [12]. All of these have in common that they contain infinitely many non-congruent pairs of tiles along a 'fault line' in the tiling. Informally, a fault line is an infinite line that separates a tiling into two halves, such that sliding the half tilings along the fault line produces tilings that still belong to the same hull. More precisely, in any given inflation tiling with an infinite fault line, there occur (countably) infinitely many distinct ways that two tiles are shifted against each other along the fault line. These shifts form a set that will have limit points. The tiling orbit closure in the local topology will then also contain tilings with shifts that correspond to these limit points, which might possibly be arbitrary real numbers.

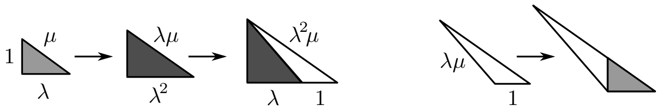
Fault lines are a typical phenomenon of non-FLC tilings. In fact, it is shown in [14] (in the proof of Thm. 4.4) that primitive stone inflations either produce FLC tilings or tilings with a fault line. Some non-trivial sufficient conditions for inflation tilings to have FLC are given in [16] and [14].

1.2. One-parameter families of inflation rules

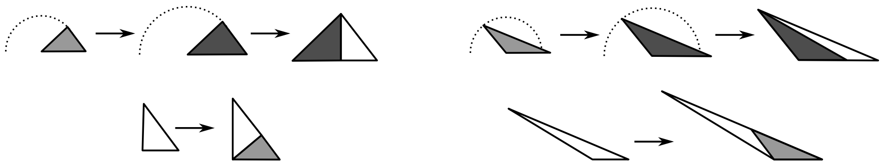
Usually, inflation rules are rigid in the sense that one cannot continuously deform the tiles without destroying the inflation property. Here, we discuss a different example of an inflation rule due to Danzer. It contains one continuous parameter which determines the shapes of the tiles. It can be found in the extended version of a paper by Goodman-Strauss [28], which is available from his website.[1]

We consider the following inflation rule for three triangular prototiles

(1.2.1)



where μ is a free parameter. Figure 1.2.1 shows a patch of a tiling arising from this inflation rule. The inflation factor $\lambda \approx 1.3247$ is the largest root of the polynomial $x^3 - x - 1$. It is the smallest Pisot–Vijayaraghavan (PV) number, sometimes called the 'plastic' number; compare [AO1, Ex. 2.17]. The value of μ can be chosen arbitrarily from the open interval $(\lambda - 1, \lambda + 1)$. Equivalently, the interior angle in the lower left vertex of the small triangle (leftmost in the inflation rule (1.2.1)) can be chosen arbitrarily from the interval $(0, \pi)$. In particular, we can produce tilings with arbitrarily 'thin' tiles in this way. The inflation rules for two further choices of μ are shown below. On the left, a realisation with three right-angled triangles is shown, while on the right the inflation uses three obtuse triangles.



Continuously decreasing or increasing the value of μ corresponds to moving the upper vertex of the first two prototiles along the half-circles indicated by dashed arcs. The upper vertex of the third prototile then moves on a different conic section. We leave it to the reader to work out the details of the latter (which is an ellipse).

[1] http://comp.uark.edu/~strauss/

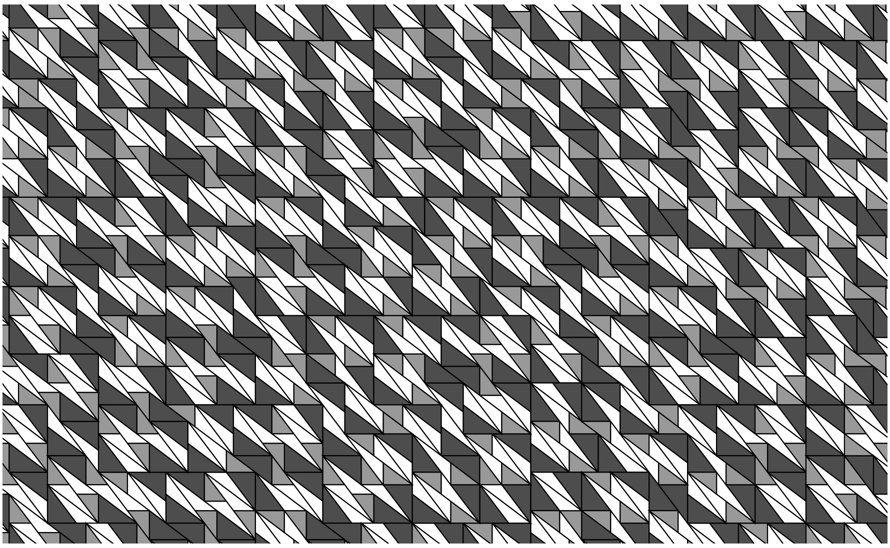
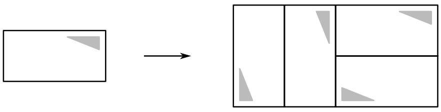


FIGURE 1.2.1. A patch of an inflation tiling generated by the inflation rule (1.2.1). Here, $\mu = \sqrt{\lambda^2 + 1}$, wherefore two of the triangles are right-angled.

1.3. A tiling with non-unique decomposition

A close relative of the table tiling (see [43] or [AO1, Ex. 6.2] for the latter) is the tiling defined by the inflation rule



If one ignores the triangular marks in the diagram, the inflated tile has less symmetry than the prototile. Hence, without the triangular marks, the diagram does *not* define an inflation uniquely. As a consequence, the tiling with unmarked tiles violates local recognisability and thus does *not* possess a *local inflation deflation symmetry* (LIDS) in the sense of [AO1, Def. 5.16]. This is indicated in the right-hand part of Figure 1.3.1.

The tiling with triangular marks does have an LIDS, as it ought to have, according to the following result by Solomyak.

Theorem 1.3.1 ([51, Thm. 1.1]). *A self-affine tiling that has FLC with respect to translations has the unique composition property if and only if it is non-periodic.* □

In our terminology, a self-affine tiling is an FLC tiling originating from a primitive stone inflation, and the unique composition property refers to

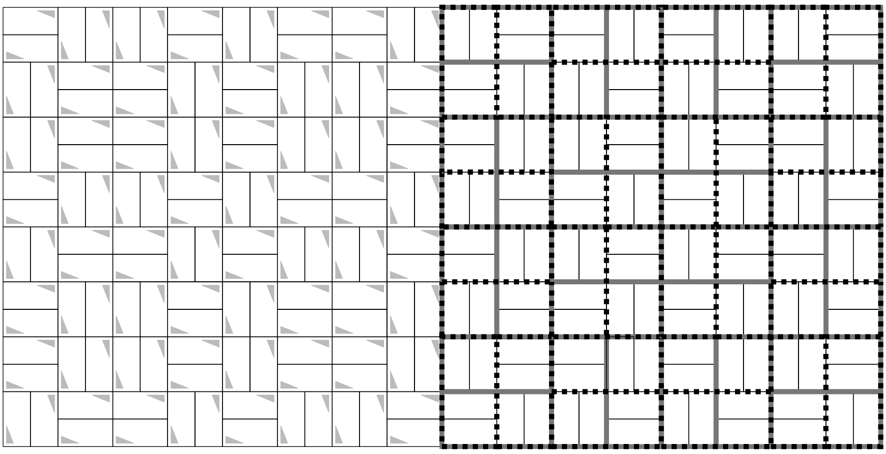


FIGURE 1.3.1. If one ignores the triangular marks in this aperiodic tiling, it has more than one possible preimage under the inflation rule. Two preimages are indicated in the right part of the figure, supertiles of one possibility with grey lines, supertiles of the other with dashed lines.

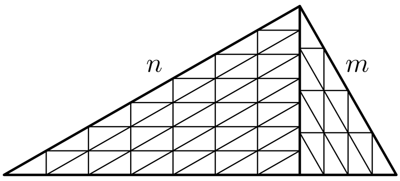
the LIDS. More precisely, the unique composition property in [51] does not require the supertiles to be determined locally. For the example at hand, this makes no difference. The tilings (marked as well as unmarked) are easily seen to be non-periodic (and hence aperiodic), either by applying Theorem 1.3.1 or by superimposing a hierarchical pattern of squares as in [AO1, Ex. 5.11]; see also [AO1, Fig. 6.50]. This example was discussed by Goodman-Strauss in [27]; see also the extended version of [28] mentioned previously.

1.4. Überpinwheel

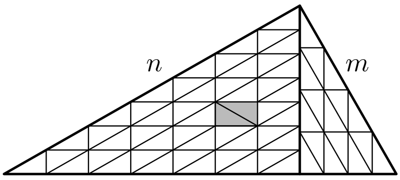
The classical *pinwheel tiling* (see [AO1, Sec. 6.6] and references therein) is an inflation tiling that fails to have FLC with respect to translations, though it has FLC with respect to rigid motions. The tiles in the pinwheel tiling are all congruent (the prototile being a right-angled triangle with edge lengths 1, 2 and $\sqrt{5}$), but they appear in (countably) infinitely many different orientations throughout the tiling. Hence, in order to specify the exact position of some tile in the pinwheel tiling, one needs three parameters with an infinite set of values rather than two; namely, two parameters for the position of its right-angled vertex, say, and one parameter in the circle $\mathbb{S}^1$ for the orientation of the tile. In the sequel, we will often identify the circle with the half-open interval $[0, 2\pi)$. The latter parameter describes the integer multiples of an irrational rotation angle, so is of the form $n\alpha \bmod 2\pi$, where $\alpha = 2\arctan(\frac{1}{2})$.

Lorenzo Sadun [45] asked whether there are planar inflation tilings that require two parameters to specify the orientation of the tiles, in the sense that there are two rationally independent, irrational rotation angles in a tiling. We are now going to discuss an example of such an 'überpinwheel' inflation tiling.

The pinwheel inflation rule is generalised as follows. Let T be a right-angled triangle with edge lengths m, n, $\sqrt{m^2+n^2} =: \lambda$, where $m, n \in \mathbb{N}$ with $m \neq n$. The classical pinwheel tiling corresponds to the case $m = 1$ and $n = 2$ (or $m = 2$ and $n = 1$). There is a canonical partition of λT into congruent copies of T:

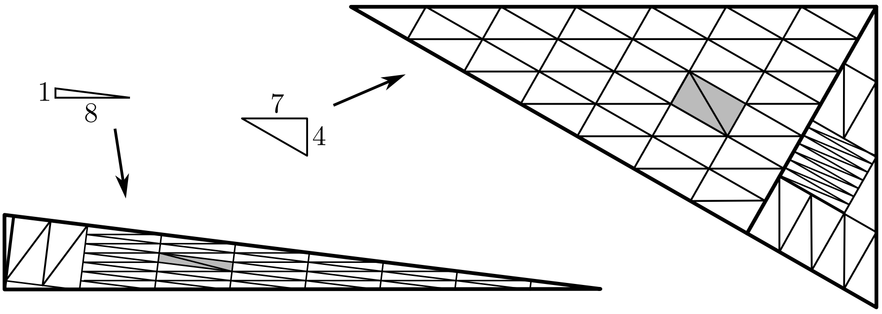


In order to define an inflation rule for an aperiodic tiling with infinitely many orientations, we need to flip at least one (but not all) of the rectangles, for instance as in



Choose an integer N such that $N = \lambda^2 = m^2 + n^2 = k^2 + \ell^2$ for $k, \ell, m, n \in \mathbb{N}$ with $m \neq n$, $k \neq \ell$ and $\{m, n\} \neq \{k, \ell\}$. Let us take the smallest choice, which is $N = 65 = 1^2 + 8^2 = 4^2 + 7^2$. Let T_1 be a right-angled triangle with edge lengths $1, 8, \sqrt{65}$ and let T_2 be a right-angled triangle with edge lengths $4, 7, \sqrt{65}$.

Consider the pinwheel-like inflation described above, applied to both triangles T_1 and T_2, but in a 'coupled' way. In order to combine these two inflations, replace a rectangular patch of size 7×8 in $\sigma(T_1)$ by a 7×8 rectangular patch of copies of T_2, and vice versa. One possible way to do so is the following:



The next result shows that each tile in the resulting tilings needs two parameters to specify its orientation.

Theorem 1.4.1. *The angles* $\arctan(\frac{1}{8})$ *and* $\arctan(\frac{4}{7})$ *are both irrational, and are independent over* $\mathbb{Q}$.

PROOF. The irrationality of $\arctan(\frac{1}{8}) = \frac{\pi}{2} - \arccos\bigl(\frac{1}{\sqrt{65}}\bigr)$ follows from the fact that $\arccos\bigl(\frac{1}{\sqrt{n}}\bigr) \notin \pi\mathbb{Q}$ for $n \geqslant 3$ odd; see for instance [1, Thm. 3]. This can be proved alternatively using cyclotomic fields. We will illustrate this with $\arctan(\frac{4}{7})$.

If $\arctan(\frac{4}{7}) \in \pi\mathbb{Q}$, then there is an $n \in \mathbb{N}$ such that $(7+4\mathrm{i})^n \in \mathbb{R}$, or equivalently there is an $n \in \mathbb{N}$ such that $\frac{7+4\mathrm{i}}{|7+4\mathrm{i}|}$ is a (complex) n-th root of unity. Then, $\frac{(7+4\mathrm{i})^2}{|7+4\mathrm{i}|^2} = \frac{7+4\mathrm{i}}{7-4\mathrm{i}}$ is also a root of unity. Since $\frac{7+4\mathrm{i}}{7-4\mathrm{i}} \in \mathbb{Q}(\mathrm{i})$, and the roots of unity in $\mathbb{Q}(\mathrm{i})$ are $\{1, \mathrm{i}, -1, -\mathrm{i}\}$ [AO1, Sec. 2.5.2], this yields a contradiction. (More generally, all roots of unity in $\mathbb{Q}(\mathrm{e}^{2\pi\mathrm{i}/n})$ are of the form $\pm\mathrm{e}^{2\pi\mathrm{i}/n}$; see [55, Exc. 2.3] or [AO1, Sec. 2.5.2].)

The independence of $\arctan(\frac{1}{8})$ and $\arctan(\frac{4}{7})$ can again be shown by interpreting them as complex numbers. If $\arctan(\frac{1}{8})$ and $\arctan(\frac{4}{7})$ were dependent over $\mathbb{Q}$, then there would exist $k, m \in \mathbb{Z} \setminus \{0\}$ such that $k \arctan(\frac{1}{8}) = m \arctan(\frac{4}{7})$. With $z := \frac{8+\mathrm{i}}{|8+\mathrm{i}|}$ and $y := \frac{7+4\mathrm{i}}{|7+4\mathrm{i}|}$, this implies that $z^k = y^m$, hence $z^{2k} = y^{2m}$, which gives

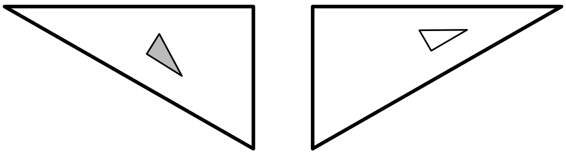
$$\frac{(8+\mathrm{i})^k}{(8-\mathrm{i})^k} = \frac{(7+4\mathrm{i})^m}{(7-4\mathrm{i})^m} \quad \text{and thus} \quad (8+\mathrm{i})^k(7-4\mathrm{i})^m = (8-\mathrm{i})^k(7+4\mathrm{i})^m.$$

Because the ring $\mathbb{Z}[\mathrm{i}]$ of Gaussian integers is a unique factorisation domain, the prime factorisation is unique up to units in $\mathbb{Z}[\mathrm{i}]$, hence

$$(-\mathrm{i})^k(1+2\mathrm{i})^k(3+2\mathrm{i})^k(1-2\mathrm{i})^m(3+2\mathrm{i})^m = \mathrm{i}^k(1-2\mathrm{i})^k(3-2\mathrm{i})^k(1+2\mathrm{i})^m(3-2\mathrm{i})^m$$

and thus $(1+2\mathrm{i})^{k-m}(3+2\mathrm{i})^{k+m} = (1-2\mathrm{i})^{k-m}(3-2\mathrm{i})^{k+m}$. Since $1+2\mathrm{i}$, $1-2\mathrm{i}$, $3+2\mathrm{i}$ and $3-2\mathrm{i}$ are pairwise coprime in $\mathbb{Z}[\mathrm{i}]$, this yields a contradiction. □

The fact that copies of both T_1 and T_2 occur in $\sigma(T_1)$ as well as in $\sigma(T_2)$ implies the primitivity of σ. Furthermore, the fact that $\sigma(T_2)$ contains two copies of T_2 that are reflected in their shortest edge ensures that the tiles in the corresponding tilings appear in infinitely many orientations. Indeed, substituting these two tiles yields two copies of T_2 that are rotated against each other by $2\arctan(\frac{4}{7})$,



Consequently, higher level supertiles contain copies of T_2 that are rotated against each other by an angle of $n \cdot 2\arctan(\frac{4}{7})$ for all $n \in \mathbb{Z}$. Since we have $\arctan(\frac{4}{7}) \notin \pi\mathbb{Q}$, these angles are distinct. In fact, whenever such a situation occurs, the angles are even uniformly distributed in $[0, 2\pi)$ by [19, Prop. 3.4 and Thm. 6.1]. This result is due to Radin [42] for the pinwheel tiling, while the general case is treated in [19].

Theorem 1.4.2 ([19, Prop. 3.4 and Thm. 6.1])**.** *Let σ be a primitive inflation rule in $\mathbb{R}^2$. Each tiling in the hull of σ has statistical circular symmetry if and only if there is a level-n supertile (for some $n \geqslant 1$) containing two copies of the same prototile which are rotated against each other by some angle $\alpha \notin \pi\mathbb{Q}$.* □

Here, statistical circular symmetry means that the orientations of the tiles are not only dense on the circle, but actually uniformly distributed. Since there are countably infinitely many orientations of tiles in the pinwheel tiling, the uniform distribution property refers to frequencies of tiles with an orientation within certain intervals. Uniform distribution then means that, for any two such intervals of the same length, the frequencies of tiles with orientations in these intervals are equal; see [AO1, Sec. 7.1] for a more precise definition.

Via similar constructions, one may obtain examples of tilings in which the orientations of tiles are described by $M > 2$ irrational angles. This can be done by mixing M pinwheel-like inflations with common inflation factor $\lambda = \sqrt{q}$, where q can be expressed as a sum of two distinct squares in M different ways. Nevertheless, illustrating these examples will be inconvenient, due to the inevitably large inflation factors. The next values are given by $325 = 1^2 + 18^2 = 6^2 + 17^2 = 10^2 + 15^2$ for $M = 3$, by $1105 = 24^2 + 23^2 = 31^2 + 12^2 = 32^2 + 9^2 = 33^2 + 4^2$ for $M = 4$, and by $5525 = 55^2 + 50^2 = 62^2 + 41^2 = 70^2 + 25^2 = 71^2 + 22^2 = 73^2 + 14^2 = 74^2 + 7^2$ for $M = 6$; compare entry A052199 in the OEIS [49].

1.5. Tile orientations with distinct frequencies

The classical pinwheel tiling and its relatives discussed above have the slightly surprising property that the tile orientations are uniformly distributed on the circle. A related result holds for tilings that have FLC with respect to translations.

Theorem 1.5.1 ([20, Thm. 2.3])**.** *Let σ be a primitive inflation rule such that the tilings in the hull of σ have FLC. If, for any two congruent tiles S and T, the patch $\sigma(S)$ is congruent to the patch $\sigma(T)$, the frequencies of congruent tiles with different orientations are equal.* □

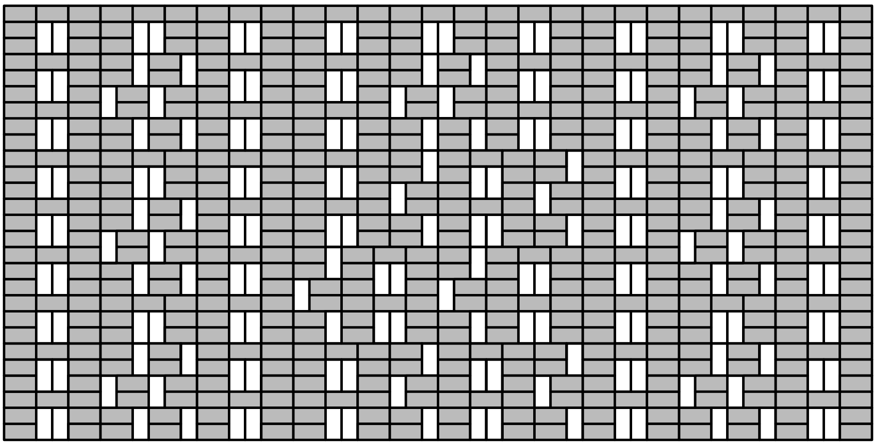
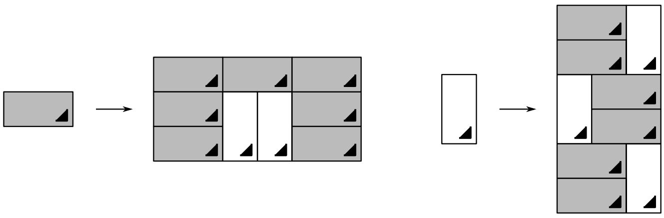


FIGURE 1.5.1. A level-3 inflation patch of the aperiodic 'punch card' tiling (without markers), in which horizontal tiles are more frequent than vertical ones. This requires different inflation rules for horizontal and vertical rectangles, as specified in Eq. (1.5.1).

Note that, throughout [AO1] and thus far in this chapter, we have tacitly taken the compatibility of inflation and rotation for granted. That is, we have implicitly assumed the condition of Theorem 1.5.1 to be satisfied. For instance, if the inflation rule of the Ammann–Beenker tiling is specified by showing the inflation of a square T as $\sigma(T)$, then we implicitly assumed that T rotated by $\pi/2$ is substituted by the patch $\sigma(T)$ rotated by $\pi/2$. However, this need not be the case in general. In order to construct a tiling where, say, horizontal rectangles are more frequent than vertical ones, one needs to specify two different inflation rules for vertical and horizontal rectangles. The following example defines such a rule,

(1.5.1)



We refer to the corresponding tilings as 'punch card' tilings. A patch is shown in Figure 1.5.1. It obviously contains more horizontal rectangles than vertical ones. More precisely, since the inflation matrix is $M_\sigma = \left(\begin{smallmatrix} 7 & 6 \\ 2 & 3 \end{smallmatrix}\right)$ with PF eigenvalue 9 and corresponding right eigenvector $(\frac{3}{4}, \frac{1}{4})^T$, there are three times as many horizontal as vertical rectangles in any tiling of the hull.