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James E. Pringle , Andrew King
Excerpt
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The basic fluid equations

The subject of this book is how the matter of the visible Universe moves. Almost all of this matter is in gaseous form, and each gram contains of order 10^{24} particles (atoms, ions, protons, electrons, etc.), all moving independently except for interactions such as collisions. At first sight it might seem an impossible task to describe the evolution of such a complicated system. However, in many cases we can avoid most of this inherent complexity by approximating the matter as a *fluid*. A fluid is an idealized continuous medium with certain macroscopic properties such as density, pressure and velocity . This concept applies equally to gases and liquids, and we shall take the term fluid to refer to both in this book. The structure of matter at the atomic or molecular level is important only in fixing relations between macroscopic fluid properties such as density and pressure, and in specifying others such as viscosity and conductivity.

Describing a medium as a fluid is possible if we can define physical quantities such as density $\rho(\mathbf{r}, \mathbf{t})$ or velocity $\mathbf{u}(\mathbf{r}, \mathbf{t})$ at a particular place with position vector $\mathbf{r}$ at time t. For a meaningful definition of a 'fluid velocity' we must average over a large number of such particles. In other words, fluid dynamical quantities are well defined only on a scale l such that l is not only much greater than a typical interparticle distance, but also, more restrictively, much greater than a typical particle mean free path, λ_{mfp}.† Further, the concept of local fluid quantities is only useful if the scale l on which they are defined is much smaller than the typical macroscopic lengthscales L on which fluid properties vary. Thus to use the equations of fluid dynamics we require $L \gg l \gg \lambda_{\mathrm{mfp}}$.

If this condition fails one should, strictly, not apply the fluid dynamical equations, but instead use concepts from plasma physics such as particle distribution functions. However, the huge additional complications and large physical uncertainties

† Roughly speaking, the mean free path is the average distance travelled by a typical particle before its trajectory is significantly deflected by another particle.

involved here mean that astrophysicists often apply fluid dynamical equations in situations where they are not strictly valid. The mean free path in astrophysical fluids is typically $\lambda_{\rm mfp} \simeq 10^6 (T^2/n)$ cm, where T is the temperature (in K) and n is the number density (in cm^{-3}). In the centre of the Sun we have $T \simeq 10^7$ K, $n \simeq 10^{26}$ cm^{-3}, so $\lambda_{\rm mfp} \sim 10^{-6}$ cm. This is far smaller than the solar radius $R_\odot = 7 \times 10^{10}$ cm, so the fluid approximation is very good. In the solar wind, however, we have $T \sim 10^5$ K, $n \sim 10$ cm^{-3} near the Earth's orbit, so that $\lambda_{\rm mfp} \sim 10^{15}$ cm. This is far greater than the Sun–Earth distance, which is 1.5×10^{13} cm. Thus the fluid approximation does not apply well here, and the treatment of the interaction of the solar wind with the Earth's magnetosphere requires plasma physics. As a final example, the diffuse gas in a cluster of galaxies typically has $T \simeq 3 \times 10^7$ K, $n \simeq 10^{-3}$ cm^{-3}, and hence $\lambda_{\rm mfp} \sim 10^{24}$ cm. This is of the same order as the physical size ~ 1 Mpc of a rich cluster. The fluid approximation is at best marginal for the diffuse regions of the cluster gas, but is nevertheless often used to gain a crude insight into its dynamics, heating and cooling. The dimensionless ratio $\lambda_{\rm mfp}/L$ of mean free path to typical flow lengthscale is called the Knudsen number Kn; $Kn \ll 1$ is a necessary condition for the validity of the fluid approximation. The results above show that $Kn \ll 1$ in the interior of the Sun, $Kn \gg 1$ in the solar wind, and $Kn \sim 1$ in cluster gas.

In this book we assume that the reader already has some familiarity with fluid dynamics, though not necessarily in an astrophysical context. For this reason the following derivation and discussion of the equations of fluid dynamics is brief. It is aimed mainly at establishing notation, as well as stressing those properties of fluids relevant to astrophysics which may be less familiar to fluid dynamicists from other fields.

1.1 Conservation of mass and momentum

The equations of fluid dynamics express conservation laws, and indeed one can use this basic property advantageously in devising numerical methods to solve them.

1.1.1 Mass conservation

Consider a fixed finite volume V within the fluid, bounded by the surface S. Then the mass of fluid contained within the volume is given by

$$\int_V \rho \, \mathrm{d}V. \tag{1.1}$$

The mass contained in V can change only through a flux of fluid through the surface S. Thus conservation of mass implies the following:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \rho \, \mathrm{d}V = - \int_S \rho \mathbf{u} \cdot \mathbf{dS}, \tag{1.2}$$

where **dS** is the (vector) element of area on the surface S. The volume is fixed, so we can take the derivative inside the term on the left-hand side (l.h.s.) and apply the divergence theorem to the term on the right-hand side (r.h.s.) to obtain

$$\int_V \left\{ \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) \right\} \mathrm{d}V = 0. \tag{1.3}$$

Since the volume V is arbitrary, we conclude that the integrand must itself vanish, that is

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{1.4}$$

and, equivalently, in suffix notation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0. \tag{1.5}$$

1.1.2 Momentum conservation

The momentum equation is obtained in exactly the same way by considering the rate of change of the total momentum in the volume V, given by

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \rho \, \mathbf{u} \, \mathrm{d}V. \tag{1.6}$$

The additional complication here is that as well as considering the flux of momentum across the surface S, we must take account of both the body force per unit volume f_i acting on the fluid and the surface stress given by an appropriate stress tensor T_{ij}. The momentum equation is then given by

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = f_i + \frac{\partial}{\partial x_j}[T_{ij}]. \tag{1.7}$$

In this book we consider two main contributors to the body force. First we write the gravitational force as follows:

$$f_i = -\rho \frac{\partial \Phi}{\partial x_i}, \tag{1.8}$$

where the gravitational potential Φ is related to the density through Poisson's equation:

$$\nabla^2 \Phi = 4\pi G \rho, \tag{1.9}$$

where G is the gravitational constant. Second we take the magnetic force in the following form:

$$f_i = (\mathbf{j} \wedge \mathbf{B})_i, \tag{1.10}$$

where $\mathbf{j}$ is the current and $\mathbf{B}$ is the magnetic field.

We shall also briefly consider the electric force,

$$f_i = \rho_Q E_i, \tag{1.11}$$

where ρ_Q is the electric charge density and **E** is the electric field.

We define the stress tensor as follows. Consider an infinitesimal vector surface element **dS** within the fluid, where by convention the magnitude of the vector is the area of the surface element and the direction of the vector is normal to the surface element. Then the surface element is subject to a surface force **F** given by

$$F_i = T_{ij}\, \mathrm{d}S_j. \tag{1.12}$$

We note that since both **dS** and **F** are vectors, then by the quotient rule T_{ij} is a second-order tensor.

In this book the main contributor to the stress tensor that we consider is the pressure p in the form

$$T_{ij} = -p\delta_{ij}, \tag{1.13}$$

where we make use of the Kronecker delta. In Section 1.5 we shall also write the magnetic force as a stress tensor as follows:

$$m_{ij} = B_i B_j - \frac{1}{2}\delta_{ij} B_k B_k. \tag{1.14}$$

Although we do not consider viscous effects in this book, we note here that the viscous stress terms come from relating the viscous contribution to the stress tensor to the second-order tensor $\partial u_i/\partial x_j$. This contains information about the relative flow of neighbouring fluid elements and is called the (rate of) strain tensor. Physically this expresses the fact that microscopic (especially thermal) motions within the ensemble of gas particles can transport momentum over distances of order the mean free path.

Finally, using the mass conservation equation, eq. (1.4), to replace the term $\partial\rho/\partial t$, we obtain the momentum equation (or the equation of motion of the fluid) in the following form:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} + \frac{\partial m_{ij}}{\partial x_j}. \tag{1.15}$$

1.2 The Lagrangian derivative

We can consider the evolution of a fluid quantity like the density $\rho(\mathbf{r}, t)$ in two ways. The partial derivative $\partial\rho/\partial t$ used above measures the way ρ changes with time t at a fixed position **r**. But it is often more useful to consider the rate of change of the density of a particular fluid element as it moves with the fluid. This rate is called the Lagrangian derivative and is denoted by $\mathrm{D}\rho/\mathrm{D}t$. We need to establish the relationship between these two concepts.

Suppose that a particular fluid element is at position $\mathbf{r}_0$ at time $t = 0$, and at a later time t is at a new position $\mathbf{r}(\mathbf{r}_0, t)$. Then the velocity of the fluid element is given by

$$\mathbf{u} = \frac{\partial}{\partial t}\mathbf{r}(\mathbf{r}_0, t), \tag{1.16}$$

where the partial derivative is taken at fixed $\mathbf{r}_0$. The Lagrangian derivative of (for example) the density of that particular fluid element is then simply given by

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\partial}{\partial t}\rho(\mathbf{r}(\mathbf{r}_0, t), t), \tag{1.17}$$

with the partial derivative taken at fixed $\mathbf{r}_0$. Since t appears in two places on the r.h.s. we may expect two terms in the derivative. Using the chain rule and the definition of $\mathbf{u}$ above we obtain

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\partial\rho}{\partial t} + \mathbf{u}\cdot\nabla\rho. \tag{1.18}$$

Thus, more generally the operator denoting the rate of change of a quantity following the fluid motion (the Lagrangian derivative) is given by

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla. \tag{1.19}$$

1.3 Conservation of energy

We consider the heat content of a unit mass of fluid. In terms of thermodynamic quantities, a small change in the internal heat content of this unit mass is given by

$$T\,\mathrm{d}S = \mathrm{d}e + p\,\mathrm{d}V, \tag{1.20}$$

where T is the temperature, S is the entropy per unit mass, e is the internal energy per unit mass and V is the volume per unit mass. In terms of the density it is evident that $V = 1/\rho$, and thus

$$T\mathrm{d}S = \mathrm{d}e - p\frac{\mathrm{d}\rho}{\rho^2}. \tag{1.21}$$

Hence in a fluid flow, the rate of change of the heat content of a particular fluid element of unit mass is given by

$$T\frac{\mathrm{D}S}{\mathrm{D}t} = \frac{\mathrm{D}e}{\mathrm{D}t} - \frac{p}{\rho^2}\frac{\mathrm{D}\rho}{\mathrm{D}t}. \tag{1.22}$$

The heat content of a fluid element can change through effects of two types.

First, there may be heat flow into or out of the element. We shall refer to this generically as ‘conduction’. However, in the astrophysical context heat can be conducted both by gas particles (typically electrons, since they move faster than the ions) as in standard thermal conduction and also by photons (known as radiative transfer). In both cases, the heat flux $\mathbf{h}$ in units of energy per unit area per unit time can often be written in the following form:

$$\mathbf{h} = -\lambda \nabla \mathbf{T}, \tag{1.23}$$

which implies physically that the heat flux occurs down the temperature gradient at a rate proportional to some 'thermal conductivity' λ. We expect λ to be a function of thermodynamic variables such as T and ρ. This form of the heat flux is appropriate provided that the particles carrying the heat have mean free paths much smaller than the typical lengthscale L over which macroscopic fluid quantities change. For electrons or molecules this is equivalent to the requirements of the fluid approximation, whereas for photons it requires in addition that the fluid should be opaque ('optically thick') so that there are very large numbers of interactions between photons and the fluid over lengthscales L.

Second, there may be internal generation of heat. This can result from dissipation of kinetic energy by viscosity or dissipation of magnetic energy through resistivity (or electrical conductivity). We do not consider these processes in this book. In the astrophysical context internal energy can be generated by nuclear processes (such as nuclear energy generation in stars) and by a change in ionization of the fluid. It can also be caused by heat exchange with particles which have a low collision cross section, for example heating by cosmic rays in the interstellar medium and radiative heating and/or cooling in an optically thin gas. We shall denote the generation of internal energy by ϵ in units of energy per unit volume per unit time.

To convert from the rate of change of a unit mass of fluid (given by eq. (1.22)) to the rate of change per unit volume, we multiply by the mass per unit volume, i.e. the density. Thus the heat equation becomes

$$\rho T \frac{\mathrm{D}S}{\mathrm{D}t} = -\mathrm{div}\,\mathbf{h} + \epsilon. \tag{1.24}$$

1.4 The equation of state and useful approximations

To complete the set of equations obtained so far we need a relationship of the form $p = p(\rho, T)$, which is the equation of state for the fluid. In this book we shall assume the simplest form of the relationship, namely the equation of state of a perfect gas,

$$p = \frac{\mathcal{R}}{\mu} \rho T, \tag{1.25}$$

where $\mathcal{R}$ is the gas constant and μ is the mean particle mass, assumed to be constant. We also note that

$$\frac{\mathcal{R}}{\mu} = c_p - c_V, \tag{1.26}$$

where $c_p = T(\partial S/\partial T)_p$ is the specific heat at constant pressure and $c_V = T(\partial S/\partial T)_V$ is the specific heat at constant volume. Alternatively this may be written as follows:

$$p = (\gamma - 1)\rho e, \tag{1.27}$$

where $\gamma = c_p/c_V$ is the ratio of specific heats, and we note for a perfect gas that

$$e = c_V T. \tag{1.28}$$

To understand the physics of a particular fluid dynamical situation it is often not necessary to include the full thermodynamic complexity of the fluid. In these cases we can simplify and/or circumvent the heat equation.

1.4.1 Incompressible approximation

The major difference between astrophysical fluids and those encountered in many terrestrial situations (including those encountered in many courses on fluid dynamics) is that astrophysical ones are highly compressible. However, in situations where fluid motions are slow compared with the sound speed, density gradients are quickly smoothed out and it is a useful approximation to treat the fluid as if it were incompressible. In physical terms this means that any particular element of the fluid does not change its density, which implies that

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0. \tag{1.29}$$

It is important to realise that this does not imply that the fluid itself has constant density, so we may **not** write ρ = constant, unless the original fluid state has uniform density.

1.4.2 Adiabatic flow

If the flow occurs fast enough that no fluid element has time to exchange heat with its surroundings, and if energy generation within the fluid is negligible, the heat equation simplifies to

$$\frac{\mathrm{D}S}{\mathrm{D}t} = 0. \tag{1.30}$$

In other words, each fluid element evolves at constant entropy – it remains on the same adiabat.

At constant entropy we note that

$$\frac{\mathrm{D}p}{\mathrm{D}t} = \left(\frac{\partial p}{\partial \rho}\right)_S \frac{\mathrm{D}\rho}{\mathrm{D}t}, \tag{1.31}$$

and that

$$\left(\frac{\partial p}{\partial \rho}\right)_S = \frac{c_p}{c_V}\left(\frac{\partial p}{\partial \rho}\right)_T. \tag{1.32}$$

Since for a perfect gas

$$\left(\frac{\partial p}{\partial \rho}\right)_T = \frac{p}{\rho}, \tag{1.33}$$

on using $\gamma = c_p/c_V$ we obtain

$$\frac{\mathrm{D}}{\mathrm{D}t}\ln p = \gamma \frac{\mathrm{D}}{\mathrm{D}t}\ln \rho. \tag{1.34}$$

Thus for adiabatic flow we may assume that

$$\frac{\mathrm{D}}{\mathrm{D}t}(p/\rho^\gamma) = 0. \tag{1.35}$$

We note again that this does not imply that the entropy of the fluid is constant everywhere. But in this case if the fluid is initially isentropic (has uniform entropy) then it remains so.

1.4.3 Barotropic flow

We can avoid using the heat equation, and therefore simplify the analysis, by assuming that pressure is solely a function of density, i.e. $p = p(\rho)$. This is a useful approximation when the detailed thermal properties of the fluid are not directly relevant to the dynamics under consideration. Barotropic flow is more general than isentropic flow, and includes isothermal flow (for which $p \propto \rho$) as well as the polytropic approximation to the equation of state (relevant to fully degenerate matter),

$$p = A\rho^{1+1/n}, \tag{1.36}$$

where A and n are constants and n is called the polytropic index.

1.5 The MHD approximation

Astrophysical fluids are usually highly ionized (and so highly conducting) and permeated by magnetic fields. Understanding the interaction between the fluid and the magnetic fields it contains is therefore often important. The usual treatment of this interaction uses the magnetohydrodynamics (MHD) approximation. We stress that this is an approximation and that, in common with the fluid approximation, it is often tempting to use it in contexts where its validity is stretched.

We start by considering a fluid flow with a typical flow lengthscale L and typical flow timescale T. The usual MHD approximation depends on the assumption that the resulting typical flow velocity U is much less than the speed of light, i.e.

$U \sim L/T \ll c$. The approximation stems from the use of Ohm's law applied locally in the frame of the fluid. Thus we need to be able to transform between the fields $(\mathbf{E}, \mathbf{B})$ in the inertial frame and the fields $(\mathbf{E}', \mathbf{B}')$ in the frame of the fluid, which is moving with velocity $\mathbf{u}$. These are related by the usual Lorentz transformation:

$$\mathbf{E}' = (1 - \gamma)\left(\frac{\mathbf{u} \cdot \mathbf{E}}{u^2}\right)\mathbf{u} + \gamma(\mathbf{E} + \mathbf{u} \wedge \mathbf{B}), \tag{1.37}$$

and

$$\mathbf{B}' = (1 - \gamma)\left(\frac{\mathbf{u} \cdot \mathbf{B}}{u^2}\right)\mathbf{u} + \gamma\left(\mathbf{B} - \frac{1}{c^2}\mathbf{u} \wedge \mathbf{E}\right), \tag{1.38}$$

where

$$\gamma = \left(1 - \frac{u^2}{c^2}\right)^{-1/2}. \tag{1.39}$$

Taking the low-velocity approximation $u^2 \ll c^2$ and neglecting terms of order (u^2/c^2), these relations become

$$\mathbf{E}' = \mathbf{E} + \mathbf{u} \wedge \mathbf{B} \tag{1.40}$$

and

$$\mathbf{B}' = \mathbf{B}. \tag{1.41}$$

The time evolution of the magnetic field is determined from the Maxwell equation,

$$\frac{\partial \mathbf{B}}{\partial t} = -\mathrm{curl}\,\mathbf{E}. \tag{1.42}$$

By comparing dimensional quantities on each side of the equation we see that to order of magnitude $B/T \sim E/L$, or equivalently $E \sim (L/T)B \sim UB$.

The second relevant Maxwell equation is as follows:

$$\mu_0^{-1}\mathrm{curl}\,\mathbf{B} = \mathbf{j} + \epsilon_0\frac{\partial \mathbf{E}}{\partial t}. \tag{1.43}$$

The second term on the r.h.s. is the displacement current, which permits the propagation of electromagnetic waves in vacuum with speed c, where $c^2 = 1/\epsilon_0\mu_0$. However, in the MHD approximation we neglect the displacement current. This is because the ratio between the displacement current and the term on the l.h.s. is given to order of magnitude as $(\epsilon_0 E/T)/(B/\mu_0 L) \sim (E/B)(U/c^2) \sim U^2/c^2 \ll 1$. Thus in the MHD approximation, electromagnetic waves are excluded and the current is given by

$$\mathbf{j} = \mu_0^{-1}\mathrm{curl}\,\mathbf{B}. \tag{1.44}$$

Since $\mathbf{B}' = \mathbf{B}$, it follows that the current in the frame of the fluid is given by

$$\mathbf{j}' = \mathbf{j}. \tag{1.45}$$

In the frame of the fluid Ohm's law becomes $\mathbf{j}' = \sigma \mathbf{E}'$, where σ is the conductivity. In this book we make the additional assumption that the conductivity is infinite, which then implies that $\mathbf{E}' = 0$, i.e. that

$$\mathbf{E} = -\mathbf{u} \wedge \mathbf{B}. \tag{1.46}$$

Substituting this into eq. (1.42) we obtain the induction equation,

$$\frac{\partial \mathbf{B}}{\partial t} = \mathrm{curl}(\mathbf{u} \wedge \mathbf{B}), \tag{1.47}$$

which describes the time evolution of the magnetic field in the ideal MHD approximation.

We also need to consider the electromagnetic force acting on the fluid. The Lorentz force is given by

$$\mathbf{f} = \rho_Q \mathbf{E} + \mathbf{j} \wedge \mathbf{B}. \tag{1.48}$$

The charge density ρ_Q is related to the electric field $\mathbf{E}$ through the following Maxwell equation:

$$\mathrm{div}\, \mathbf{E} = \rho_Q / \epsilon_0. \tag{1.49}$$

Thus the ratio between the electric and magnetic contributions to the Lorentz force on the fluid is (using eq. (1.44)) to order of magnitude $(\epsilon_0 E^2/L)/(B^2/L\mu_0) \sim U^2/c^2$. Further, the current $\rho_Q \mathbf{u}$ supplied by the moving charge density is also $\sim U^2/c^2$ times the current $\mathbf{j}$. Thus in the MHD approximation we can neglect both the electric charge and the electric field, and the electromagnetic force on the fluid is (using eq. (1.44)) simply given by

$$\mathbf{f} = \mu_0^{-1} (\mathrm{curl}\, \mathbf{B} \wedge \mathbf{B}). \tag{1.50}$$

We can write this as

$$f_i = \frac{\partial m_{ik}}{\partial x_k}, \tag{1.51}$$

where

$$m_{ik} = \mu_0^{-1} \left(B_i B_k - \frac{1}{2} B^2 \delta_{ik} \right), \tag{1.52}$$

and we have used the final Maxwell equation,

$$\mathrm{div}\, \mathbf{B} = 0. \tag{1.53}$$

1.5.1 Notation and units

We can now see that in the MHD approximation the electric field does not appear in any of the equations. The magnetic field appears only in the induction equation and in the Lorentz force. The induction equation is already dimensionally consistent and so does not change if different units are used for $\mathbf{B}$. In the Lorentz force the