

LONDON MATHEMATICAL SOCIETY LECTURE NOTE SERIES

Managing Editor:

Professor N. J. Hitchin, Mathematical Institute, University of Oxford, 24-29 St Giles,
 Oxford OX1 3LB, United Kingdom

The titles below are available from booksellers, or from Cambridge University Press at
www.cambridge.org/mathematics

- 215 Number theory 1992–93, S. DAVID (ed)
- 216 Stochastic partial differential equations, A. ETHERIDGE (ed)
- 217 Quadratic forms with applications to algebraic geometry and topology, A. PFISTER
- 218 Surveys in combinatorics, 1995, P. ROWLINSON (ed)
- 220 Algebraic set theory, A. JOYAL & I. MOERDIJK
- 221 Harmonic approximation., S.J. GARDINER
- 222 Advances in linear logic, J.-Y. GIRARD, Y. LAFONT & L. REGNIER (eds)
- 223 Analytic semigroups and semilinear initial boundary value problems, KAZUAKI TAIRA
- 224 Computability, enumerability, unsolvability, S.B. COOPER, T.A. SLAMAN & S.S. WAINER (eds)
- 225 A mathematical introduction to string theory, S. ALBEVERIO, *et al*
- 226 Novikov conjectures, index theorems and rigidity I, S. FERRY, A. RANICKI & J. ROSENBERG (eds)
- 227 Novikov conjectures, index theorems and rigidity II, S. FERRY, A. RANICKI & J. ROSENBERG (eds)
- 228 Ergodic theory of Z^d actions, M. POLLICOTT & K. SCHMIDT (eds)
- 229 Ergodicity for infinite dimensional systems, G. DA PRATO & J. ZABCZYK
- 230 Prolegomena to a middlebrow arithmetic of curves of genus 2, J.W.S. CASSELS & E.V. FLYNN
- 231 Semigroup theory and its applications, K.H. HOFMANN & M.W. MISLOVE (eds)
- 232 The descriptive set theory of Polish group actions, H. BECKER & A.S. KECHRIS
- 233 Finite fields and applications, S. COHEN & H. NIEDERREITER (eds)
- 234 Introduction to subfactors, V. JONES & V.S. SUNDER
- 235 Number theory 1993–94, S. DAVID (ed)
- 236 The James forest, H. FETTER & B. G. DE BUEN
- 237 Sieve methods, exponential sums, and their applications in number theory, G.R.H. GREAVES *et al*
- 238 Representation theory and algebraic geometry, A. MARTSINKOVSKY & G. TODOROV (eds)
- 240 Stable groups, F.O. WAGNER
- 241 Surveys in combinatorics, 1997, R.A. BAILEY (ed)
- 242 Geometric Galois actions I, L. SCHNEPS & P. LOCHAK (eds)
- 243 Geometric Galois actions II, L. SCHNEPS & P. LOCHAK (eds)
- 244 Model theory of groups and automorphism groups, D. EVANS (ed)
- 245 Geometry, combinatorial designs and related structures, J.W.P. HIRSCHFELD *et al*
- 246 p -Automorphisms of finite p -groups, E.I. KHUKHRO
- 247 Analytic number theory, Y. MOTOHASHI (ed)
- 248 Tame topology and o-minimal structures, L. VAN DEN DRIES
- 249 The atlas of finite groups: ten years on, R. CURTIS & R. WILSON (eds)
- 250 Characters and blocks of finite groups, G. NAVARRO
- 251 Gröbner bases and applications, B. BUCHBERGER & F. WINKLER (eds)
- 252 Geometry and cohomology in group theory, P. KROPHOLLER, G. NIBLO & R. STÖHR (eds)
- 253 The q -Schur algebra, S. DONKIN
- 254 Galois representations in arithmetic algebraic geometry, A.J. SCHOLL & R.L. TAYLOR (eds)
- 255 Symmetries and integrability of difference equations, P.A. CLARKSON & F.W. NIJHOFF (eds)
- 256 Aspects of Galois theory, H. VÖLKLEIN *et al*
- 257 An introduction to noncommutative differential geometry and its physical applications 2ed,
 J. MADORE
- 258 Sets and proofs, S.B. COOPER & J. TRUSS (eds)
- 259 Models and computability, S.B. COOPER & J. TRUSS (eds)
- 260 Groups St Andrews 1997 in Bath, I, C.M. CAMPBELL *et al*
- 261 Groups St Andrews 1997 in Bath, II, C.M. CAMPBELL *et al*
- 262 Analysis and logic, C.W. HENSON, J. IOVINO, A.S. KECHRIS & E. ODELL
- 263 Singularity theory, B. BRUCE & D. MOND (eds)
- 264 New trends in algebraic geometry, K. HULEK, F. CATANESE, C. PETERS & M. REID (eds)
- 265 Elliptic curves in cryptography, I. BLAKE, G. SEROUSSI & N. SMART
- 267 Surveys in combinatorics, 1999, J.D. LAMB & D.A. PREECE (eds)
- 268 Spectral asymptotics in the semi-classical limit, M. DIMASSI & J. SJÖSTRAND
- 269 Ergodic theory and topological dynamics, M.B. BEKKA & M. MAYER
- 270 Analysis on Lie groups, N.T. VAROPOULOS & S. MUSTAPHA
- 271 Singular perturbations of differential operators, S. ALBEVERIO & P. KURASOV
- 272 Character theory for the odd order theorem, T. PETERFALVI
- 273 Spectral theory and geometry, E.B. DAVIES & Y. SAFAROV (eds)
- 274 The Mandelbrot set, theme and variations, TAN LEI (ed)
- 275 Descriptive set theory and dynamical systems, M. FOREMAN *et al*
- 276 Singularities of plane curves, E. CASAS-ALVERO

- 277 Computational and geometric aspects of modern algebra, M.D. ATKINSON *et al*
 278 Global attractors in abstract parabolic problems, J.W. CHOLEWA & T. DLOTKO
 279 Topics in symbolic dynamics and applications, F. BLANCHARD, A. MAASS & A. NOGUEIRA (eds)
 280 Characters and automorphism groups of compact Riemann surfaces, T. BREUER
 281 Explicit birational geometry of 3-folds, A. CORTI & M. REID (eds)
 282 Auslander-Buchweitz approximations of equivariant modules, M. HASHIMOTO
 283 Nonlinear elasticity, Y. FU & R.W. OGDEN (eds)
 284 Foundations of computational mathematics, R. DEVORE, A. ISERLES & E. SÜLI (eds)
 285 Rational points on curves over finite fields, H. NIEDERREITER & C. XING
 286 Clifford algebras and spinors 2ed, P. LOUNESTO
 287 Topics on Riemann surfaces and Fuchsian groups, E. BUJALANCE *et al*
 288 Surveys in combinatorics, 2001, J. HIRSCHFELD (ed)
 289 Aspects of Sobolev-type inequalities, L. SALOFF-COSTE
 290 Quantum groups and Lie theory, A. PRESSLEY (ed)
 291 Tits buildings and the model theory of groups, K. TENT (ed)
 292 A quantum groups primer, S. MAJID
 293 Second order partial differential equations in Hilbert spaces, G. DA PRATO & J. ZABCZYK
 294 Introduction to the theory of operator spaces, G. PISIER
 295 Geometry and Integrability, L. MASON & YAVUZ NUTKU (eds)
 296 Lectures on invariant theory, I. DOLGACHEV
 297 The homotopy category of simply connected 4-manifolds, H.-J. BAUES
 298 Higher operads, higher categories, T. LEINSTER
 299 Kleinian Groups and Hyperbolic 3-Manifolds Y. KOMORI, V. MARKOVIC & C. SERIES (eds)
 300 Introduction to Möbius Differential Geometry, U. HERTRICH-JEROMIN
 301 Stable Modules and the D(2)-Problem, F.E.A. JOHNSON
 302 Discrete and Continuous Nonlinear Schrödinger Systems, M. J. ABLORWITZ, B. PRINARI & A. D. TRUBATCH
 303 Number Theory and Algebraic Geometry, M. REID & A. SKOROBOGATOV (eds)
 304 Groups St Andrews 2001 in Oxford Vol. 1, C.M. CAMPBELL, E.F. ROBERTSON & G.C. SMITH (eds)
 305 Groups St Andrews 2001 in Oxford Vol. 2, C.M. CAMPBELL, E.F. ROBERTSON & G.C. SMITH (eds)
 306 Peyresq lectures on geometric mechanics and symmetry, J. MONTALDI & T. RATIU (eds)
 307 Surveys in Combinatorics 2003, C. D. WENSLEY (ed.)
 308 Topology, geometry and quantum field theory, U. L. TILLMANN (ed)
 309 Corings and Comodules, T. BRZEZINSKI & R. WISBAUER
 310 Topics in Dynamics and Ergodic Theory, S. BEZUGLYI & S. KOLYADA (eds)
 311 Groups: topological, combinatorial and arithmetic aspects, T. W. MÜLLER (ed)
 312 Foundations of Computational Mathematics, Minneapolis 2002, FELIPE CUCKER *et al* (eds)
 313 Transcendental aspects of algebraic cycles, S. MÜLLER-STACH & C. PETERS (eds)
 314 Spectral generalizations of line graphs, D. CVETKOVIC, P. ROWLINSON & S. SIMIC
 315 Structured ring spectra, A. BAKER & B. RICHTER (eds)
 316 Linear Logic in Computer Science, T. EHRHARD *et al* (eds)
 317 Advances in elliptic curve cryptography, I. F. BLAKE, G. SEROUSSI & N. SMART
 318 Perturbation of the boundary in boundary-value problems of Partial Differential Equations, D. HENRY
 319 Double Affine Hecke Algebras, I. CHEREDNIK
 321 Surveys in Modern Mathematics, V. PRASOLOV & Y. ILYASHENKO (eds)
 322 Recent perspectives in random matrix theory and number theory, F. MEZZADRI & N. C. SNAITH (eds)
 323 Poisson geometry, deformation quantisation and group representations, S. GUTT *et al* (eds)
 324 Singularities and Computer Algebra, C. LOSSEN & G. PFISTER (eds)
 325 Lectures on the Ricci Flow, P. TOPPING
 326 Modular Representations of Finite Groups of Lie Type, J. E. HUMPHREYS
 328 Fundamentals of Hyperbolic Manifolds, R. D. CANARY, A. MARDEN & D. B. A. EPSTEIN (eds)
 329 Spaces of Kleinian Groups, Y. MINSKY, M. SAKUMA & C. SERIES (eds)
 330 Noncommutative Localization in Algebra and Topology, A. RANICKI (ed)
 331 Foundations of Computational Mathematics, Santander 2005, L. PARDO, A. PINKUS, E. SULI & M. TODD (eds)
 332 Handbook of Tilting Theory, L. ANGELERI HÜGEL, D. HAPPEL & H. KRAUSE (eds)
 333 Synthetic Differential Geometry 2ed, A. KOCK
 334 The Navier-Stokes Equations, P. G. DRAZIN & N. RILEY
 335 Lectures on the Combinatorics of Free Probability, A. NICU & R. SPEICHER
 336 Integral Closure of Ideals, Rings, and Modules, I. SWANSON & C. HUNEKE
 337 Methods in Banach Space Theory, J. M. F. CASTILLO & W. B. JOHNSON (eds)
 338 Surveys in Geometry and Number Theory, N. YOUNG (ed)
 339 Groups St Andrews 2005 Vol. 1, C.M. CAMPBELL, M. R. QUICK, E.F. ROBERTSON & G.C. SMITH (eds)
 340 Groups St Andrews 2005 Vol. 2, C.M. CAMPBELL, M. R. QUICK, E.F. ROBERTSON & G.C. SMITH (eds)
 341 Ranks of Elliptic Curves and Random Matrix Theory, J. B. CONREY, D. W. FARMER, F. MEZZADRI & N. C. SNAITH (eds)

Cambridge University Press
978-0-521-86891-4 - Lectures on Kahler Geometry
Andrei Moroianu
Frontmatter
[More information](#)

Lectures on Kähler Geometry

ANDREI MOROIANU
École Polytechnique, Paris



Cambridge University Press
978-0-521-86891-4 - Lectures on Kahler Geometry
Andrei Moroianu
Frontmatter
[More information](#)

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521868914

© A. Moroianu 2007

This publication is in copyright. Subject to statutory exception
and to the provisions of relevant collective licensing agreements,
no reproduction of any part may take place without
the written permission of Cambridge University Press.

First published 2007

Printed in the United Kingdom at the University Press, Cambridge

A catalog record for this publication is available from the British Library

ISBN 978-0-521-86891-4 hardback
ISBN 978-0-521-68897-0 paperback

Cambridge University Press has no responsibility for the persistence or accuracy of URLs for
external or third-party internet websites referred to in this publication, and does not guarantee
that any content on such websites is, or will remain, accurate or appropriate.

Contents

Introduction	ix
Part 1. Basics of differential geometry	1
Chapter 1. Smooth manifolds	3
1.1. Introduction	3
1.2. The tangent space	4
1.3. Vector fields	6
1.4. Exercises	9
Chapter 2. Tensor fields on smooth manifolds	13
2.1. Exterior and tensor algebras	13
2.2. Tensor fields	15
2.3. Lie derivative of tensors	17
2.4. Exercises	19
Chapter 3. The exterior derivative	21
3.1. Exterior forms	21
3.2. The exterior derivative	21
3.3. The Cartan formula	23
3.4. Integration	24
3.5. Exercises	26
Chapter 4. Principal and vector bundles	29
4.1. Lie groups	29
4.2. Principal bundles	31
4.3. Vector bundles	33
4.4. Correspondence between principal and vector bundles	33
4.5. Exercises	35
Chapter 5. Connections	37
5.1. Covariant derivatives on vector bundles	37
5.2. Connections on principal bundles	39
5.3. Linear connections	41
5.4. Pull-back of bundles	41
5.5. Parallel transport	42
5.6. Holonomy	43
5.7. Reduction of connections	44

vi	CONTENTS	
5.8.	Exercises	45
Chapter 6.	Riemannian manifolds	47
6.1.	Riemannian metrics	47
6.2.	The Levi–Civita connection	48
6.3.	The curvature tensor	49
6.4.	Killing vector fields	51
6.5.	Exercises	52
Part 2.	Complex and Hermitian geometry	55
Chapter 7.	Complex structures and holomorphic maps	57
7.1.	Preliminaries	57
7.2.	Holomorphic functions	59
7.3.	Complex manifolds	59
7.4.	The complexified tangent bundle	61
7.5.	Exercises	62
Chapter 8.	Holomorphic forms and vector fields	65
8.1.	Decomposition of the (complexified) exterior bundle	65
8.2.	Holomorphic objects on complex manifolds	67
8.3.	Exercises	68
Chapter 9.	Complex and holomorphic vector bundles	71
9.1.	Holomorphic vector bundles	71
9.2.	Holomorphic structures	72
9.3.	The canonical bundle of $\mathbb{C}P^m$	74
9.4.	Exercises	75
Chapter 10.	Hermitian bundles	77
10.1.	The curvature operator of a connection	77
10.2.	Hermitian structures and connections	78
10.3.	Exercises	80
Chapter 11.	Hermitian and Kähler metrics	81
11.1.	Hermitian metrics	81
11.2.	Kähler metrics	82
11.3.	Characterization of Kähler metrics	83
11.4.	Comparison of the Levi–Civita and Chern connections	85
11.5.	Exercises	86
Chapter 12.	The curvature tensor of Kähler manifolds	87
12.1.	The Kählerian curvature tensor	87
12.2.	The curvature tensor in local coordinates	88
12.3.	Exercises	91
Chapter 13.	Examples of Kähler metrics	93

CONTENTS

vii

13.1.	The flat metric on \mathbb{C}^m	93
13.2.	The Fubini–Study metric on the complex projective space	93
13.3.	Geometrical properties of the Fubini–Study metric	95
13.4.	Exercises	97
Chapter 14.	Natural operators on Riemannian and Kähler manifolds	99
14.1.	The formal adjoint of a linear differential operator	99
14.2.	The Laplace operator on Riemannian manifolds	100
14.3.	The Laplace operator on Kähler manifolds	101
14.4.	Exercises	104
Chapter 15.	Hodge and Dolbeault theories	105
15.1.	Hodge theory	105
15.2.	Dolbeault theory	107
15.3.	Exercises	109
Part 3.	Topics on compact Kähler manifolds	111
Chapter 16.	Chern classes	113
16.1.	Chern–Weil theory	113
16.2.	Properties of the first Chern class	116
16.3.	Exercises	118
Chapter 17.	The Ricci form of Kähler manifolds	119
17.1.	Kähler metrics as geometric U_m -structures	119
17.2.	The Ricci form as curvature form on the canonical bundle	119
17.3.	Ricci-flat Kähler manifolds	121
17.4.	Exercises	122
Chapter 18.	The Calabi–Yau theorem	125
18.1.	An overview	125
18.2.	Exercises	127
Chapter 19.	Kähler–Einstein metrics	129
19.1.	The Aubin–Yau theorem	129
19.2.	Holomorphic vector fields on Kähler–Einstein manifolds	131
19.3.	Exercises	133
Chapter 20.	Weitzenböck techniques	135
20.1.	The Weitzenböck formula	135
20.2.	Vanishing results on Kähler manifolds	137
20.3.	Exercises	139
Chapter 21.	The Hirzebruch–Riemann–Roch formula	141
21.1.	Positive line bundles	141
21.2.	The Hirzebruch–Riemann–Roch formula	142
21.3.	Exercises	145

Chapter 22. Further vanishing results	147
22.1. The Lichnerowicz formula for Kähler manifolds	147
22.2. The Kodaira vanishing theorem	149
22.3. Exercises	151
Chapter 23. Ricci-flat Kähler metrics	153
23.1. Hyperkähler manifolds	153
23.2. Projective manifolds	155
23.3. Exercises	156
Chapter 24. Explicit examples of Calabi–Yau manifolds	159
24.1. Divisors	159
24.2. Line bundles and divisors	161
24.3. Adjunction formulas	162
24.4. Exercises	165
Bibliography	167
Index	169

Introduction

These notes grew out of my graduate course at Hamburg University in the autumn of 2003. Their main purpose is to provide a quick and modern introduction to different aspects of Kähler geometry. I had tried to make the original lectures accessible to graduate students in mathematics and theoretical physics having only basic knowledge of calculus in several variables and linear algebra. The present notes should (hopefully) have retained this quality.

The text is organized as follows. The first part is devoted to a review of basic differential geometry. We discuss here topics related to smooth manifolds, tensors, Lie groups, principal bundles, vector bundles, connections, holonomy groups, Riemannian metrics, and Killing vector fields.

The reader familiar with the contents of a first course in differential geometry can pass directly to the second part, which starts with a description of complex manifolds and holomorphic vector bundles. Kähler manifolds are then discussed from the point of view of Riemannian geometry. This part ends with an outline of Hodge and Dolbeault theories, and a simple proof of the famous Kähler identities.

In the third part we study several aspects of compact Kähler manifolds: the Calabi conjecture, Weitzenböck techniques, Calabi–Yau manifolds, and divisors.

The material contained in each chapter is equivalent to a ninety-minute lecture. All chapters end with a series of exercises. Solving them may prove to be at least helpful, if not sufficient, for a reasonable understanding of the theory.

Acknowledgements. I would like to thank Christian Bär and the Department of Mathematics of Hamburg University for having invited me to teach this graduate course. During the preparation of the manuscript I had many discussions with Paul Gauduchon and Uwe Semmelmann which helped me a lot to improve the presentation. I am also indebted to Mihaela Pilca and Liviu Ornea for their critical reading of a preliminary version of these notes. Finally, I would like to thank Roger Astley for his advice and his moral support.

Paris, September 2006