Market-Valuation Methods in Life and Pension Insurance

In classical life insurance mathematics, the obligations of the insurance company towards the policy holders were calculated on artificial conservative assumptions on mortality and interest rates. However, the classical approach is being superseded by developments in international accounting and solvency standards coupled with theoretical advances in the understanding of the principles and methods for a more market-based valuation of risk, i.e. its price if traded in a free market.

The book describes these new approaches, and is the first to explain them in conjunction with more traditional methods. The exposition integrates methods and results from financial and insurance mathematics, and is based on the entries in a life insurance company's market accounting scheme. With-profit insurance contracts are described in a classical actuarial model with a deterministic interest rate and no investment alternatives. The classical valuation based on conservative valuation assumptions is explained and an alternative market-valuation approach is introduced and generalized to stochastic interest rates and risky investment alternatives. The problem of incompleteness in insurance markets is addressed using a variety of methods, for example risk minimization, mean-variance hedging and utility optimization. The application of mathematical finance to unit-linked life insurance is unified with the theory of distribution of surplus in life and pension insurance. The final chapter provides an introduction to interest rate derivatives and their use in life insurance.

The book will be of great interest and use to students and practitioners who need an introduction to this area, and who seek a practical yet sound guide to life insurance accounting and product development.

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Preface

Insurance mathematics and financial mathematics have converged during the last few decades of the twentieth century and this convergence is expected to continue in the future. New valuation methods are added to the traditional valuation methods of insurance mathematics. Valuation and decision making on the asset side and the liability side of the insurance companies are, to an increasing extent, being considered as two sides of the same story.

The development has two consequences. Demands are made on practising actuaries, whose education dates back to when financial mathematics was not considered as an integrated part of insurance mathematics. By considering the convergence as it applies to their daily work, such actuaries should be kept abreast of this convergence. From this starting point, the ideas, concepts and results of finance should be brought together to construct a path between classical actuarial deterministic patterns of thinking and modern actuarial mathematics. This is where stochastic processes are brought to the surface in payment streams as well as in investment possibilities.

At the same time, present students of actuarial mathematics need to apply financial mathematics to classical insurance valuation problems. These students will typically, and should, meet financial mathematics in textbooks on pure finance. However, to receive the full benefit of financial mathematical skills, these skills need to be integrated and proven beneficial for classical problems of insurance mathematics already on a student level.

International accounting standards have developed over the years. Denmark has been at the forefront, implementing new accounting methods to replace (assumed to be) conservative book values with real values based on market information. Although the international accounting standards have not yet been settled, the Danish approach to market valuation seems to be an important step in the right direction. Many aspects of this approach are underpinned by methods taken from mathematical finance.

The rationale for this book is that practising actuaries need an exposition of financial methods and their applications to life insurance from the point of view of a practitioner. Methods and applications are discussed in terms of the Danish approach to market valuation. As a by-product, the book explains to present students how financial methods known to them can be applied to valuation problems in the life insurance market.

In 1995 and 1996, Tomas Björk and Ragnar Norberg gave courses in financial mathematics and applications to life insurance at the University of Copenhagen. These courses aroused our interest in the interplay between finance and insurance. We studied the topics in our master theses, finished in 1996 (T.M.) and 1997 (M.S.), and continued our studies in our Ph.D. theses, finished in 2000 and 2001, respectively.

Parts of the book (Chapter 2–5) are based on material which was developed for a course on market valuation in life and pension insurance. This course was organized by the Danish Actuarial Association in 2001. Each chapter was the material for one course module and was written more or less independently of the others. The material was originally written in Danish. In 2002 the material was developed further and translated into English, and Chapter 1 was added. In 2003, Chapters 6 and 7 were added on the occasion of The First Nordic summer school in insurance mathematics, entitled "New Financial Products in Insurance." In 2004, the material was made consistent for notation and terminology. However, it is still our intention that each chapter should be readable more or less independently of the others. Therefore, some definitions and introductions of quantities are repeated throughout the book.

The book suggests approaches to life insurance market-valuation problems. The starting point is the version of with-profit insurance provided by Danish life insurance companies. In order to help the reader follow the mathematical description of this type of product, Chapter 1, written by Mogens Steffensen, provides a non-mathematical introduction to life insurance practise in general.

In Chapter 2, also written by Mogens Steffensen, the with-profit insurance contract is described at first in a classical actuarial model with a deterministic interest rate and no investment alternatives. The classical valuation based on conservative valuation assumptions is explained, and an alternative market-valuation approach is introduced. Here, the partition of future payments in guaranteed payments and non-guaranteed payments is important. Particular attention is paid to the intervention options held by the policy holder, i.e. the surrender and free policy (paid-up policy) options. Various approaches to these options are suggested.

The market-valuation method introduced in Chapter 2 is generalized to a stochastic interest rate in Chapter 3. Both discrete-time and continuous-time

bond market theory are introduced to a level such that the reader can follow the reasoning behind replacing the discount factor in the market-valuation formulas for guaranteed payments by zero coupon bond prices. Fundamental financial concepts, such as arbitrage and market completeness, are introduced in a bond market framework. Difference and differential equations for the market value of guaranteed payments are derived. Chapter 3 is written by Thomas Møller.

In Chapter 4 the market-valuation method introduced in Chapter 2 is generalized to a situation with one risky investment alternative to the deterministic interest rate. Both discrete-time and continuous-time stock market theory are introduced to a level such that the reader can follow calculations of market valuations of non-guaranteed payments in the case of two investment alternatives. Fundamental financial concepts, such as arbitrage and market completeness, are repeated in a stock market framework. Difference and differential equations for the market value of the total payments, including the non-guaranteed payments, are derived. Finally, the stock market is connected to the stochastic bond market introduced in Chapter 3. Chapter 4 is written by Mogens Steffensen.

The usual outline of introductory financial mathematics is to introduce the fundamental financial concepts in a discrete and/or continuous stock market model. Afterwards these are repeated in a discrete- and/or continuous-time bond market model. In our exposition, the cut is different and is based on entries in a life insurance company's market accounting scheme. Valuating the guaranteed payments, the important stochastic generalization of the classical actuarial deterministic financial market lies at the introduction of a stochastic bond market. A stochastic stock market comes into play when valuating the non-guaranteed payments.

An alternative class of insurance contracts to with-profit insurance is unit-linked insurance contracts, which are studied in Chapter 5, written by Thomas Møller. This class of contracts and their market values are analyzed on the basis of the stochastic financial markets introduced in Chapters 3 and 4. The problem of genuine incompleteness in insurance markets is addressed. The incompleteness in insurance markets stemming, for example, from mortality risk is often taken care of in the literature by assuming risk neutrality of the insurance company with respect to such a risk. Relaxing this assumption, we suggest various approaches to incomplete market valuation. In particular, valuation and optimal investment methods based on risk minimization, mean-variance hedging and utility optimization are introduced and exemplified in the case of unit-linked insurance.

In Chapter 6, the application of mathematical finance to unit-linked life insurance is unified with the theory of distribution of surplus in life and pension insurance. The unification is based on a consideration of distribution of surplus as an integrated part of the insurance contract. The notion of surplus and various dividend and bonus schemes linked to this surplus are studied. In particular, explicit results are obtained in the case where dividends and bonus payments are linear in the surplus. Chapter 6 is written by Mogens Steffensen.

Typically, insurance companies are faced by insurance liabilities that extend up to sixty years into the future, whereas the financial markets typically do not offer bonds that extend more than thirty years into the future. With market-based valuation methods, the value of both assets and liabilities are affected by changes in the economic environment. Here, interest rate derivatives seem to have become an important risk-management tool for life insurance companies. An introduction to certain concepts and instruments from the area of interest rate derivatives is therefore given in Chapter 7, written by Thomas Møller. Examples are swap rates, swaps, floors, caps, swaptions and CMS options. Various pricing methods are discussed, and it is demonstrated how the financial impact on a life insurance company of these instruments could be assessed.

The book studies approaches for market valuation of life insurance liabilities. The various chapters address specific aspects of market-based valuation and contain introductions to theoretical results from financial mathematics and stochastic calculus that are necessary for the applications. A brief discussion of the relation to existing books on financial mathematics and insurance mathematics is given in the following list.

- Björk, T. (2004). *Arbitrage Theory in Continuous Time*, 2nd edn (Oxford: Oxford University Press). Most of the theoretical results related to financial mathematics presented here can be found in this book by Tomas Björk. To some extent, the notation suggested by Björk is considered as "standard" and is therefore used in the present book. Even the structural exposition of certain topics of financial mathematics is inspired by Björk's book.
- Briys, E. and de Varenne, F. (2001). *Insurance: From Underwriting to Derivatives: Asset Liability Management in Insurance Companies* (Chichester, UK: Wiley). The book discusses the convergence between the insurance industry and the capital markets. It is less mathematical than the current book and focuses on institutional aspects of the interplay between the two fields. In contrast, the current book investigates the convergence of the theories of financial mathematics and insurance mathematics and their applications to market-based valuation.

- Gerber, H.U. (1997). *Life Insurance Mathematics* (Berlin: Springer). This provides an introduction to classical life insurance mathematics and can be viewed as a necessary prerequisite for the current book. The concepts and techniques discussed in Gerber's book (for example, present values, specific life insurance contracts, decrement series, Thiele's differential equation) are also used and explained in the present book. However, the current manuscript has a completely different goal and goes considerably beyond the introductory presentation in Gerber's book.
- Koller, M. (2000). *Stochastische Modelle in der Lebensversicherung* (Berlin: Springer). The book presents a framework where the underlying insurance contracts are modeled by Markov chains and where stochastic interest rates are allowed. The main difference between Koller's book and the current manuscript is that we focus more on market values and the application of theories from financial mathematics in the area of life insurance; Koller's book deals more with the underlying Markov chains and on deriving differential equations for the corresponding reserves (based on the work by Hoem, Norberg and others).

We expect our readership to fall into two categories. Firstly, practising life insurance actuaries who need an update of the mathematics of life insurance, an introduction to financial mathematics in an insurance context and an approach to market valuation in life insurance. Indeed, the book takes the point of view of a practising actuary. Chapter 1 on life insurance practice will provide the reader with sufficient insight into this practice.

Secondly, it is expected that the book will be read by students in actuarial science, who have prerequisites in both life insurance mathematics and mathematical finance, but want to see how these disciplines can and will be combined in both theory and practice. By taking the viewpoint of a practising actuary as a starting point, the student also sees how aspects of the classical life insurance mathematics are implemented in practice.

The level is advanced. Basic knowledge of life insurance mathematics (such as in Gerber's book) is required. In addition, basic probability theory is required, such that the reader can follow the introduction of filtrations, martingales, stochastic differential equations, etc. No previous knowledge of financial mathematics is required. However, the theoretical results are at times developed quickly in order to get to the applications, and at some points the reader would probably benefit from studying textbooks on financial mathematics for more details and more background information; for example, Björk (2004).

We would like to thank Tomas Björk and Ragnar Norberg for arousing our curiosity and for sharpening our understanding of the mathematics of life

insurance and finance. The combination of basic knowledge in both areas and a provoked curiosity made our studies of the interplay between these fields possible, challenging and interesting. In addition, we wish to thank Ragnar Norberg, Christian Hipp and Martin Schweizer for their guidance and support during our Ph.D. studies. We would also like to thank Vibeke Thinggard and Mikkel Jarbøl for valuable comments and discussions on earlier versions of this material: in 2001 they were members of the Continued Professional Development Committee under the Danish Actuarial Association and were deeply involved in the organization of the course given in 2001.