Introduction

Fragmentation and coagulation are two natural phenomena that can be observed in many sciences, at a great variety of scales. To give just a few examples, let us simply mention first for fragmentation, the studies of stellar fragments in astrophysics, fractures and earthquakes in geophysics, breaking of crystals in crystallography, degradation of large polymer chains in chemistry, DNA fragmentation in biology, fission of atoms in nuclear physics, fragmentation of a hard drive in computer science, ... For coagulation, we mention the formation of the large structures (galaxies) in the universe and of planets by accretion in astrophysics, of polymer chains in chemistry, of droplets of liquids in aerosols or clouds, coalescence of ancestral lineages in genealogy of populations in genetics, ...

The main purpose of this monograph is to develop mathematical models which may be used in situations where either phenomenon occurs randomly and repeatedly as time passes. For instance, in the case of fragmentation, we can think of the evolution of blocks of mineral in a crusher. The text is intended for readers having a solid background in probability theory.[1] I aimed at providing a rather concise and self-contained presentation of random fragmentation and coagulation processes; I endeavored to make accessible some recent developments in this field, but did not try to be exhaustive. Each chapter ends with a 'Comments' section in which some important aspects that have not been discussed in the main part of the text (often because the discussion would have been too technical and/or lengthy) are addressed and precise references are given.

[1] Preliminaries on classical properties of some fundamental stochastic processes (including Markov chains in continuous time, Poisson random measures, ...) will be provided in the text; however, the reader is expected to be already acquainted with basic notions in this area.

Let us first briefly discuss fragmentation. In order to deal with models that can be studied mathematically, we are led to make hypotheses which may look at first sight somewhat stringent, but that are however, commonly assumed in applications. First, we suppose that the system has a memoryless evolution, that is its future only depends on its present state and not on its past. In particular, this excludes the possibility that an object might be more fragile (i.e. more likely to split) due to former shocks. Second, we assume that each fragment can be characterized by a real number that should be thought of as its size. This stops us from considering the spatial position of a fragment or further geometrical properties like its shape; physicists call such models *mean field*. Finally, we shall always suppose that the evolution of a given fragment does not depend on its environment, in the sense that fragments split independently of each other, or in other words, that the branching property is fulfilled.

Similarly, the coalescent processes that we shall consider here are particle systems with a Markovian (i.e. memoryless) evolution, where particles are characterized by their sizes. Roughly, the key assumption is that coagulations occur at rates which only depend on the particles involved in the merging, and not on the other particles in the system. Very loosely speaking, this hypothesis plays a role similar to the branching property for fragmentation processes.

Naively, fragmentation and coagulation are dual notions, in the sense that a simple time-reversal changes one into the other. However, due to the fundamental requirements we impose (these are needed to get notions that can be investigated mathematically), this naive duality relation fails for the fragmentation and coalescent processes that are considered in this text. This means that in general, time-reversal does not transform a fragmentation process into a coalescent process, and vice-versa.[2] Therefore we will have to develop distinct theories. However, they share similarities, not just because they deal with random processes taking values in the same state space but more significantly because the same techniques are often relevant to investigate both notions.

The plan of this text is as follows. In the first chapter, we develop the theory of fragmentation chains, that is fragmentation processes in which each fragment remains stable for some random time and then splits. We focus on processes having a natural property of self-similarity. The heart of the study lies in the underlying genealogical structure of the fragmentation chain as a randomly marked tree, and the branching property. We investigate asymptotic properties of the empirical measure of the fragments as time tends to infinity;

[2] Nonetheless, there are a few striking and important examples in which the duality holds, that will be emphasized in this text.

the so-called intrinsic martingale induced by the genealogical tree plays a fundamental role in the study. When the index of self-similarity is negative, we point at unexpected phenomena of extinction and formation of dust.

Plainly, this discrete approach does not enable us to consider situations where splitting can occur continuously, that is when each fragment may split immediately. We shall make a key step to solve this fundamental difficulty in the second chapter. There, we will start by discussing various notions of partitions, which provide natural frameworks for fragmentation and coalescent processes. One obvious notion is that of a partition of a unit mass, that is a sequence of non-negative real numbers which add up to at most 1. An important class of random mass-partitions can be constructed from Poisson random measures, and we shall develop some material in this field, including the classical Poisson-Dirichlet distributions and its two-parameter generalization by Pitman and Yor. A probably less intuitive notion, which is due to Kingman, is that of an exchangeable random partition of $\mathbb{N}$, the set of natural integers. Roughly, the latter arises by sampling points at random in the object which splits, and provides a powerful technique of spatial discretization for mass-partitions.

Exchangeable random partitions play a key role in Chapter 3 for the construction and study of general self-similar fragmentation processes, in which each fragment may split immediately. We shall characterize their dynamics in terms of an erosion coefficient, a rate of sudden dislocation, and the index of self-similarity. The evolution of a fragment containing a point tagged at random can then be described in terms of some subordinator (i.e. an increasing process with independent and stationary increments) whose characteristics are expressed in terms of the erosion coefficient, the dislocation rate, and the index of self-similarity. Statistical properties of the tagged fragment yield extensions of results proved for fragmentation chains in the first chapter to this more general setting.

Coalescent processes are considered in the last two chapters; we shall focus on two quite different types of dynamics. The first are the so-called exchangeable coalescents for which the rates of coagulation do not depend on the masses in the system. In general, coagulations may occur simultaneously, and each coagulation may involve an arbitrary number of components. The second type comprises the so-called stochastic coalescents, which are often used as simple models for random coagulation in physics. There, only binary coagulations are permitted, but now the rate of such a coagulation may depend on the two fragments which are involved.

Exchangeable coalescents are studied in Chapter 4. We first introduce the celebrated coalescent of Kingman (the essentially unique exchangeable

coalescent having only binary coagulations), which plays an important role in the study of the genealogy of large populations. Then, we shall consider general exchangeable coalescents, which have been introduced by Pitman, Möhle, Sagitov and Schweinsberg. The analyses often rely on the same techniques and ideas as those which have been used for homogeneous fragmentations, even though time-reversing a homogeneous fragmentation does not produce an exchangeable coalescent as one might expect naively. Finally, we develop connections with certain stochastic flows, following a recent work by Bertoin and Le Gall, and investigate the special case of the so-called Bolthausen-Sznitman coalescent which was motivated by considerations in statistical physics.

Finally, Chapter 5 is devoted to the stochastic coalescents of Marcus and Lushnikov, and their asymptotics. We shall pay special attention to the problem of the regularity of the transition probabilities as a function of the initial conditions, following a beautiful approach due to Fournier. We shall also develop the connection with Smoluchowski's coagulation equations, by presenting some results about the hydrodynamic behavior of certain stochastic coalescents. In this direction, we shall first investigate the multiplicative coalescent via its representation in terms of the Erdős-Rényi random graph model, and then extend the analysis to sub-multiplicative kernels by coupling. The ultimate section of this chapter will deal with the additive kernel, focussing on remarkable connections with certain random structures (random trees and forests) which have been developed by Pitman.

It should be already clear from the brief presentation above that this text owes much to Jim Pitman. Many of his ideas, results and intuitions form the cornerstones of this study. The reader is strongly encouraged to read his superb Lecture Notes [186] for the St Flour summer school, which covers in particular several topics treated here, sometimes from a different perspective. Needless to say that my warmest thanks go to Jim.

Certain parts presented in this text result directly from collaborations. I would like to express my deep gratitude to Sasha Gnedin, Christina Goldschmidt, Jean-François Le Gall, Servet Martinez, Jim Pitman and Alain Rouault. Working with them in this field has been not only enriching and stimulating, but also very pleasant.

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ideas they developed, have been for me a major source of motivation and support for carrying on this project.

Further, I would like to thank Maria-Emilia Caballero, Pablo Ferrari and Servet Martinez, who offered me the opportunity of giving short courses on fragmentation and coagulation. Earlier drafts of certain portions of this text have been used as unpublished lecture notes corresponding to these courses.

Last but not least, this text has been written while I had a position at the Institut universitaire de France; I would probably not have found time nor energy for undertaking this project without the support of this institution.

1
Self-similar fragmentation chains

Informally, imagine an object that falls apart randomly as time passes. The state of the system at some given time consists in the sequence of the sizes of the pieces, which are often called fragments or particles. Suppose that the evolution is Markovian and obeys the following rules. First, different particles evolve independently of each other, that is the so-called branching property is fulfilled. Second, there is a parameter $\alpha \in \mathbb{R}$, which will be referred to as the index of self-similarity, such that each fragment with size s is stable during an exponential time with parameter proportional to s^α. In other words, a particle with size $s > 0$ has an exponential lifetime with mean $cs^{-\alpha}$, where $c > 0$ is some constant. At its death, this particle splits and there results a family of fragments, say with sizes $(s_i, i \in \mathbb{N})$, where the sequence of ratios $(s_i/s, i \in \mathbb{N})$ has the same distribution for all particles. The purpose of this chapter is to construct such self-similar fragmentation chains, to shed light on their genealogical structure, and to establish some of their fundamental properties.

1.1 Construction of fragmentation chains

In this section, we briefly present some basic elements on Markov chains and branching Markov chains in continuous time which are then used for the construction and the study of fragmentation chains. For convenience, we recall first some standard notation for sets of integers which will be used through this text without further reference.

The sets of positive integers, and respectively of integers, are denoted by

$$\mathbb{N} = \{1, 2, \dots\}\,, \quad \mathbb{Z} = \{\dots, -1, 0, 1, \dots\}\,,$$

and then

$$\mathbb{Z}_+ = \mathbb{N} \cup \{0\} = \{0, 1, 2, \dots\}$$

designates the set of non-negative integers. When we shall need to consider infinity as an extended integer, we shall use the notation

$$\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}\,, \quad \overline{\mathbb{Z}}_+ = \mathbb{Z}_+ \cup \{\infty\}\,.$$

1.1.1 Preliminaries on Markov chains

Let (E, d) be a Polish space, that is a complete separable metric space, which we also endow with its Borel sigma-field. Consider a collection $(q(x, \cdot), x \in E)$ of finite measures on E which is (weakly) measurable in the variable x, in the sense that for every Borel set $B \subseteq E$, the map $x \to q(x, B)$ is measurable. It is well-known that we can use the kernel $(q(x, \cdot), x \in E)$ as the jump rates of some Markov chain in continuous time. Let us briefly recall the main steps and refer, for example, to Norris [174] or Fristedt and Gray [109] for more details.

For every $x \in E$, we write $q(x) := q(x, E)$ for the total mass of the measure $q(x, \cdot)$, and we introduce the normalized probability measure on E given by

$$\bar{q}(x, \cdot) = q(x, \cdot)/q(x)$$

with the convention that $\bar{q}(x, \cdot) = \delta_x$ is the Dirac point mass at x when $q(x) = 0$. So $(\bar{q}(x, \cdot), x \in E)$ is a Markov kernel, that is a (weakly) measurable family of probability measures on E. We can think of $(\bar{q}(x, \cdot), x \in E)$ as the transition probabilities of a Markov sequence[1] $Y = (Y(n), n \in \mathbb{Z}_+)$. That is, for every $n \in \mathbb{Z}_+$,

$$\mathbb{P}\left(Y(n+1) \in \cdot \mid Y(0), \dots, Y(n)\right) = \bar{q}(Y(n), \cdot)\,.$$

Next, we shall transform the Markov sequence Y into a Markov process $X = (X(t), t \geq 0)$ in continuous time which visits the same states as the

[1] In the literature, Markov sequences are often called Markov chain in discrete time. However, in order to avoid a possible confusion with Markov chains in continuous time, which are the main object of interest in this section, we shall keep the terminology *chain* for processes in continuous time, and use *sequence* for processes in discrete time. We also mention that in the literature, Markov chains in continuous time generally concern only countable state spaces; some authors prefer to refer to pure-jump Markov processes in the case of a general topological state space. Nonetheless, we shall use here the name *chain* to underline the hold-jump structure (which will be described below), and keep the name *process* for continuous evolutions, that is situations where the process may not remain constant on arbitrarily small time-interval.

sequence Y. More precisely, conditionally on the sequence $Y = (y_n, n \in \mathbb{Z}_+)$, we shall replace the unit waiting time at each step y_n by an exponential variable with parameter $q(y_n)$ (thus depending on the state of Y at this step), independently of the other steps. The construction is specified by the following procedure.

Let $\mathbf{e}_0, \mathbf{e}_1, \ldots$ be a sequence of i.i.d. standard exponential variables, which is independent of Y. We associate to every sample path of Y the additive functional

$$A(n) := \sum_{i=0}^{n} \mathbf{e}_i / q(Y(i)), \qquad n \in \mathbb{Z}_+,$$

which represents the instant at which X jumps from the state $Y(n)$ to the state $Y(n+1)$. This procedure enables us to define $X(t)$ for any $t \geq 0$ if and only if the series $A(\infty) := \sum_{i=0}^{\infty} \mathbf{e}_i / q(Y(i))$ diverges. In this direction, we recall the following well-known fact.

Lemma 1.1 *The conditions*

$$A(\infty) := \sum_{i=0}^{\infty} \mathbf{e}_i / q(Y(i)) = \infty \qquad a.s. \tag{1.1}$$

and

$$\sum_{i=0}^{\infty} 1/q(Y(i)) = \infty \qquad a.s. \tag{1.2}$$

are equivalent.

Proof We shall prove a slightly stronger result. Let $(y_i, i \in \mathbb{Z}_+)$ be some sequence of points in E, which should be thought of as the sequence of the states visited by Y. On the one hand, the identity

$$\mathbb{E}\left(\sum_{i=0}^{\infty} \mathbf{e}_i / q(y_i)\right) = \sum_{i=0}^{\infty} 1/q(y_i)$$

shows that if the series on the right-hand side converges, then $\sum_{i=0}^{\infty} \mathbf{e}_i / q(y_i) < \infty$ a.s. Conversely, taking the Laplace transform, we get

$$\mathbb{E}\left(\exp\left(-\sum_{i=0}^{\infty} \mathbf{e}_i / q(y_i)\right)\right) = \prod_{i=0}^{\infty} \frac{q(y_i)}{1+q(y_i)}$$

$$= \exp\left(-\sum_{i=0}^{\infty} \ln\left(1 + 1/q(y_i)\right)\right).$$

If the series $\sum_{i=0}^{\infty} \mathbf{e}_i/q(y_i)$ converges with positive probability, then the right-hand side above has to be strictly positive and hence $\sum_{i=0}^{\infty} 1/q(y_i) < \infty$. Note from the first part of the proof that this forces the series $\sum_{i=0}^{\infty} \mathbf{e}_i/q(y_i)$ to converge with probability one, a fact that can also be observed directly from Kolmogorov's 0-1 law. □

Condition (1.2) is plainly fulfilled whenever

$$\sup_{x\in E} q(x) < \infty\,; \tag{1.3}$$

however, in general checking whether (1.2) holds can be tedious. Henceforth, taking (1.1) for granted, we may introduce the time-change

$$\alpha(t) = \min\{n \in \mathbb{Z}_+ : A(n) > t\}\,, \qquad t \geq 0;$$

one says that $\alpha(\cdot)$ is the right-continuous inverse of the additive functional $A(\cdot)$. Then we define a process in continuous time $X = (X(t), t \geq 0)$ by the identity

$$X(t) := Y(\alpha(t))\,, \qquad t \geq 0\,.$$

This construction by random time-substitution can be rephrased in terms of a so-called *hold-jump* description: the states $x \in E$ with $q(x) = 0$ are *absorbing* for X, that is

$$\mathbb{P}(X(t) = x \text{ for all } t \geq 0 \mid X(0) = x) = 1\,,$$

and starting from some non-absorbing state $x \in E$ with $q(x) > 0$, the process X stays at x up to the holding time $\mathbf{e}_0/q(x)$ which has an exponential distribution with parameter $q(x)$, and then jumps[2] according to the probability distribution $\bar{q}(x, \cdot)$, independently of the holding time. It is easily seen from the absence of memory of exponential variables that X enjoys the Markov property; one says that X is a Markov chain (in continuous time). Note also that X has right-continuous paths a.s.

The semigroup $(P_t, t \geq 0)$ of X is the family of linear operators on the space of bounded measurable functions $f : E \to \mathbb{R}$ defined by

$$P_t f(x) := \mathbb{E}(f(X(t) \mid X(0) = x)\,, \qquad x \in E;$$

[2] When $q(x, \{x\}) > 0$, the probability that process X stays at the state x after the exponential holding time is positive, so strictly speaking there may be no jump after this first holding time. However, this induces no difficulty whatsoever, and it is convenient not to distinguish this degenerate case.

it satisfies the *Chapman-Kolmogorov equation*

$$P_t \circ P_s = P_{t+s}\,, \qquad t, s \geq 0\,.$$

It is easy to check from the hold-jump description that for every bounded measurable function $f : E \to \mathbb{R}$,

$$\begin{aligned} \mathtt{G}f(x) &:= \lim_{t\to 0+} \frac{1}{t}\mathbb{E}(f(X(t)) - f(X(0)) \mid X(0) = x) \\ &= \int_E (f(y) - f(x))\, q(x, dy)\,, \end{aligned} \tag{1.4}$$

which identifies the infinitesimal generator $\mathtt{G}$ of X. In particular, combining with the Chapman-Kolmogorov equation yields the classical *backward* equation

$$\frac{dP_t f(x)}{dt} = \mathtt{G}P_t f(x)\,, \qquad t \geq 0\,. \tag{1.5}$$

Further, when the function $\mathtt{G}f$ is bounded on E, we also have the *forward* equation

$$\frac{dP_t f(x)}{dt} = P_t \mathtt{G} f(x)\,, \qquad t \geq 0\,. \tag{1.6}$$

A well-known alternative characterization of the infinitesimal generator is that for every bounded measurable function $f : E \to \mathbb{R}$ such that $\mathtt{G}f$ is bounded, $\mathtt{G}f$ is the unique bounded measurable function $g : E \to \mathbb{R}$ for which the process

$$f(X(t)) - \int_0^t g(X(s))ds\,, \qquad t \geq 0$$

is a martingale under $\mathbb{P}(\cdot \mid X(0) = x)$ for every $x \in E$.

Either the construction of X or (1.4), shows that the family $(q(x, \cdot), x \in E)$ can be thought of as the *jump rates* of X, and thus entirely characterizes the distribution of the Markov chain X. In the same vein, note also that when the space E is discrete and $q(x, \{x\}) = 0$, the jump rates of the chain can be recovered from its one-dimensional distributions by

$$q(x, \{y\}) = \lim_{t\to 0+} \frac{1}{t}\mathbb{P}(X(t) = y \mid X(0) = x)\,, \qquad x \neq y. \tag{1.7}$$

Example The so-called *compound Poisson processes* form one of the simplest and best known family of Markov chains in continuous time. Specifically, consider the special case when E is some Euclidean space (or, more generally, some nice topological group) and the jump rates $(q(x, \cdot), x \in E)$ are translation invariant, that is for every $x \in E$, $q(x, \cdot)$ is the image of some given