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1 Cartesian Tensor Analysis

1.1 Introduction

In this chapter we present an elementary introduction to Cartesian tensor analysis in a three-dimensional Euclidean point space or a two-dimensional subspace. A Euclidean point space is the space of position vectors of points. The term vector is used in the sense of classical vector analysis, and scalars and polar vectors are zeroth- and first-order tensors, respectively. The distinction between polar and axial vectors is discussed later in this chapter. A scalar is a single quantity that possesses magnitude and does not depend on any particular coordinate system, and a vector is a quantity that possesses both magnitude and direction and has components, with respect to a particular coordinate system, which transform in a definite manner under change of coordinate system. Also vectors obey the parallelogram law of addition. There are quantities that possess both magnitude and direction but are not vectors, for example, the angle of finite rotation of a rigid body about a fixed axis.

A second-order tensor can be defined as a linear operator that operates on a vector to give another vector. That is, when a second-order tensor operates on a vector, another vector, in the same Euclidean space, is generated, and this operation can be illustrated by matrix multiplication. The components of a vector and a second-order tensor, referred to the same rectangular Cartesian coordinate system, in a three-dimensional Euclidean space, can be expressed as a (3×1) matrix and a (3×3) matrix, respectively. When a second-order tensor operates on a vector, the components of the resulting vector are given by the matrix product of the (3×3) matrix of components of the second-order tensor and the matrix of the (3×1)

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components of the original vector. These components are with respect to a rectangular Cartesian coordinate system, hence the term Cartesian tensor analysis. Examples from classical mechanics and stress analysis are as follows. The angular momentum vector, $\boldsymbol{h}$, of a rigid body about its mass center is given by $\boldsymbol{h} = \boldsymbol{J}\boldsymbol{\omega}$ where $\boldsymbol{J}$ is the inertia tensor of the body about its mass center and $\boldsymbol{\omega}$ is the angular velocity vector. In this equation the components of the vectors $\boldsymbol{h}$ and $\boldsymbol{\omega}$ can be represented by (3×1) matrices and the tensor $\boldsymbol{J}$ by a (3×3) matrix with matrix mutiplication implied. A further example is the relation $\boldsymbol{t} = \boldsymbol{\sigma}\boldsymbol{n}$, between the stress vector $\boldsymbol{t}$ acting on a material area element and the unit normal $\boldsymbol{n}$ to the element, where $\boldsymbol{\sigma}$ is the Cauchy stress tensor. The relations $\boldsymbol{h} = \boldsymbol{J}\boldsymbol{\omega}$ and $\boldsymbol{t} = \boldsymbol{\sigma}\boldsymbol{n}$ are examples of coordinate-free symbolic notation, and the corresponding matrix relations refer to a particular coodinate system.

We will meet further examples of the operator properties of second-order tensors in the study of continuum mechanics and thermodynamics.

Tensors of order greater than two can be regarded as operators operating on lower-order tensors. Components of tensors of order greater than two cannot be expressed in matrix form.

It is very important to note that physical laws are independent of any particular coordinate system. Consequently, equations describing physical laws, when referred to a particular coordinate system, must transform in definite manner under transformation of coordinate systems. This leads to the concept of a tensor, that is, a quantity that does not depend on the choice of coordinate system. The simplest tensor is a scalar, a zeroth-order tensor. A scalar is represented by a single component that is invariant under coordinate transformation. Examples of scalars are the density of a material and temperature.

Higher-order tensors have components relative to various coordinate systems, and these components transform in a definite way under transformation of coordinate systems. The velocity $\boldsymbol{v}$ of a particle is an example of a first-order tensor; henceforth we denote vectors, in symbolic notation, by lowercase bold letters. We can express $\boldsymbol{v}$ by its components relative to any convenient coordinate system, but since $\boldsymbol{v}$ has no preferential relationship to any particular coordinate system, there must be a definite relationship between components of $\boldsymbol{v}$ in different

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coordinate systems. Intuitively, a vector may be regarded as a directed line segment, in a three-dimensional Euclidean point space E_3, and the set of directed line segments in E_3, of classical vectors, is a vector space V_3. That is, a classical vector is the difference of two points in E_3. A vector, according to this concept, is a first-order tensor. A discussion of linear vector spaces is given in Appendix 4.

There are many physical laws for which a second-order tensor is an operator associating one vector with another. Remember that physical laws must be independent of a coordinate system; it is precisely this independence that motivates us to study tensors.

1.2 Rectangular Cartesian Coordinate Systems

The simplest type of coordinate system is a rectangular Cartesian system, and this system is particularly useful for developing most of the theory to be presented in this text.

A rectangular Cartesian coordinate system consists of an orthonormal basis of unit vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and a point 0 which is the origin. Right-handed Cartesian coordinate systems are considered, and the axes in the $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ directions are denoted by $0x_1$, $0x_2$, and $0x_3$, respectively, rather than the more usual $0x$, $0y$, and $0z$. A right-handed system is such that a 90° right-handed screw rotation along the $0x_1$ direction rotates $0x_2$ to $0x_3$, similarly a right-handed rotation about $0x_2$ rotates $0x_3$ to $0x_1$, and a right-handed rotation about $0x_3$ rotates $0x_1$ to $0x_2$.

A right-handed system is shown in Figure 1.1. A point, $x \in \boldsymbol{E}_3$, is given in terms of its coordinates (x_1, x_2, x_3) with respect to the coordinate system $0x_1x_2x_3$ by

$$\boldsymbol{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3,$$

which is a bound vector or position vector.

If points $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{E}_3$, $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{y}$ is a vector, that is, $\boldsymbol{u} \in \mathrm{V}_3$. The vector $\boldsymbol{u}$ is given in terms of its components (u_1, u_2, u_3), with respect to the rectangular

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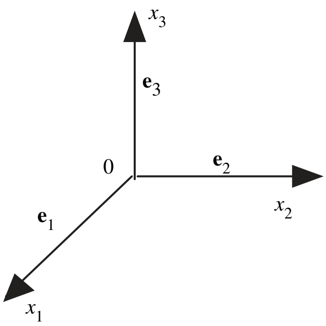

Figure 1.1. Right-handed rectangular Cartesian coordinate system.

coordinate system, $0x_1x_2x_3$ by

$$\boldsymbol{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3.$$

Henceforth in this chapter when the term coordinate system is used, a rectangular Cartesian system is understood. When the components of vectors and higher-order tensors are given with respect to a rectangular Cartesian coordinate system, the theory is known as Cartesian tensor analysis.

1.3 Suffix and Symbolic Notation

Suffixes are used to denote components of tensors, of order greater than zero, referred to a particular rectangular Cartesian coordinate system. Tensor equations can be expressed in terms of these components; this is known as suffix notation. Since a tensor is independent of any coordinate system but can be represented by its components referred to a particular coordinate system, components of a tensor must transform in a definite manner under transformation of coordinate systems. This is easily seen for a vector. In tensor analysis, involving oblique Cartesian or curvilinear coordinate systems, there is a distinction between what are called contravariant and covariant components of tensors but this distinction disappears when rectangular Cartesian coordinates are considered exclusively.

Bold lower- and uppercase letters are used for the symbolic representation of vectors and second-order tensors, respectively. Suffix notation is used to specify the components of tensors, and the convention that

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a lowercase letter suffix takes the values 1, 2, and 3 for three-dimensional and 1 and 2 for two-dimensional Euclidean spaces, unless otherwise indicated, is adopted. The number of distinct suffixes required is equal to the order of the tensor. An example is the suffix representation of a vector $\boldsymbol{u}$, with components (u_1, u_2, u_3) or $u_i, i \in \{1, 2, 3\}$. The vector is then given by

$$\boldsymbol{u} = \sum_{i=1}^{3} u_i \mathbf{e}_i. \tag{1.1}$$

It is convenient to use a summation convention for repeated letter suffixes. According to this convention, if a letter suffix occurs twice in the same term, a summation over the repeated suffix from 1 to 3 is implied without a summation sign, unless otherwise indicated. For example, equation (1.1) can be written as

$$\boldsymbol{u} = u_i \mathbf{e}_{\mathrm{i}} = u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 + u_3 \mathbf{e}_3 \tag{1.2}$$

without the summation sign. The sum of two vectors is commutative and is given by

$$\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u} = (u_i + v_i)\mathbf{e}_i,$$

which is consistent with the parallelogram rule. A further example of the summation convention is the scalar or inner product of two vectors,

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3. \tag{1.3}$$

Repeated suffixes are often called dummy suffixes since any letter that does not appear elsewhere in the expression may be used, for example,

$$u_i v_i = u_j v_j.$$

Equation (1.3) indicates that the scalar product obeys the commutative law of algebra, that is,

$$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}.$$

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The magnitude $|\boldsymbol{u}|$ of a vector $\boldsymbol{u}$ is given by

$$|\boldsymbol{u}| = \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}} = \sqrt{u_i u_i}.$$

Other examples of the use of suffix notation and the summation convention are

$$\begin{aligned} C_{ii} &= C_{11} + C_{22} + C_{33} \\ C_{ij} b_j &= C_{i1} b_1 + C_{i2} b_2 + C_{i3} b_3. \end{aligned}$$

A suffix that appears once in a term is known as a free suffix and is understood to take in turn the values 1, 2, 3 unless otherwise indicated. If a free suffix appears in any term of an equation or expression, it must appear in all the terms.

1.4 Orthogonal Transformations

The scalar products of orthogonal unit base vectors are given by

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}, \tag{1.4}$$

where δ_{ij} is known as the Kronecker delta and is defined as

$$\delta_{ij} = \begin{cases} 1 \text{ for } i = j \\ 0 \text{ for } i \neq j \end{cases}. \tag{1.5}$$

The base vectors $\mathbf{e}_i$ are orthonormal, that is, of unit magnitude and mutually perpendicular to each other. The Kronecker delta is sometimes called the substitution operator because

$$u_j \delta_{ij} = u_1 \delta_{i1} + u_2 \delta_{i2} + u_3 \delta_{i3} = u_i. \tag{1.6}$$

Consider a right-handed rectangular Cartesian coordinate system $0x'_i$ with the same origin as $0x_i$ as indicated in Figure 1.2. Henceforth, primed quantities are referred to coordinate system $0x'_i$.

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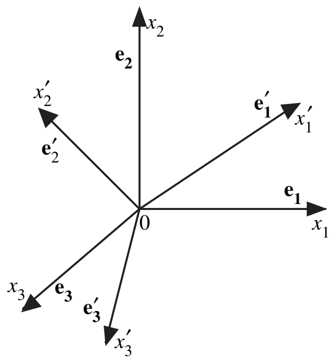

Figure 1.2. Change of axes.

The coordinates of a point P are x_i with respect to $0x_i$ and x_i' with respect to $0x_i$. Consequently,

$$x_i\mathbf{e}_i = x_j'\mathbf{e}_j', \tag{1.7}$$

where the $\mathbf{e}_i'$ are the unit base vectors for the system $0x_i$. Forming the inner product of each side of equation (1.7) with $\mathbf{e}_k'$ and using equation (1.4) and the substitution operator property equation (1.6) gives

$$x_k' = a_{ki}x_i, \tag{1.8}$$

where

$$a_{ki} = \mathbf{e}_k' \cdot \mathbf{e}_i = \cos(x_k' 0 x_i). \tag{1.9}$$

Similarly

$$x_i = a_{ki}x_k'. \tag{1.10}$$

It is evident that the direction of each axis $0x_k'$ can be specified by giving its direction cosines $a_{ki} = \mathbf{e}_k' \cdot \mathbf{e}_i = \cos(x_k' 0 x_i)$ referred to the original axes $0x_i$. The direction cosines, $a_{ki} = \mathbf{e}_k' \cdot \mathbf{e}_i$, defining this change of axes are tabulated in Table 1.1.

The matrix $[\boldsymbol{a}]$ with elements a_{ij} is known as the transformation matrix; it is not a tensor.

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Table 1.1. *Direction cosines for rotation of axes*

	$e_1{}'$	$e_2{}'$	$e_3{}'$
e_1	a_{11}	a_{21}	a_{31}
e_2	a_{12}	a_{22}	a_{32}
e_3	a_{13}	a_{23}	a_{33}

It follows from equations (1.8) and (1.10) that

$$a_{ki} = \frac{\partial x'_k}{\partial x_i} = \frac{\partial x_i}{\partial x'_k}, \tag{1.11}$$

and from equation (1.7) that

$$\mathbf{e}_{\mathrm{i}} \frac{\partial x_i}{\partial x'_k} = \mathbf{e}_{\mathrm{j}}{}' \frac{\partial x'_j}{\partial x'_k} = \mathbf{e}'_k, \tag{1.12}$$

since $\partial \boldsymbol{x}_j / \partial \boldsymbol{x}_k = \partial_{jk}$, and from equations (1.11) and (1.12) that

$$\mathbf{e}'_k = a_{ki}\mathbf{e}_i, \tag{1.13}$$

and

$$\mathbf{e}_i = a_{ki}\mathbf{e}'_k. \tag{1.14}$$

Equations (1.13) and (1.14) are the transformation rules for base vectors. The nine elements of a_{ij} are not all independent, and in general,

$$a_{ki} \neq a_{ik}.$$

A relation similar to equations (1.8) and (1.10),

$$u'_k = a_{ki}u_i, \text{ and } u_i = a_{ki}u'_k \tag{1.15}$$

is obtained for a vector $\boldsymbol{u}$ since $u_i\mathbf{e}_i = u'_k\mathbf{e}'_k$, which is similar to equation (1.7) except that the u_i are the components of a vector and the x_i are coordinates of a point.

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The magnitude $|\boldsymbol{u}| = (u_i u_i)^{1/2}$ of the vector $\boldsymbol{u}$ is independent of the orientation of the coordinate system, that is, it is a scalar invariant; consequently,

$$u_i u_i = u'_k u'_k. \tag{1.16}$$

Eliminating u_i from equation (1.15) gives

$$u'_k = a_{ki} a_{ji} u'_j,$$

and since $u'_k = \delta_{kj} u'_j$,

$$a_{ki} a_{ji} = \delta_{kj}. \tag{1.17}$$

Similarly, eliminating u_k from equation (1.15) gives

$$a_{ik} a_{jk} = \delta_{ij}. \tag{1.18}$$

It follows from equation (1.17) or (1.18) that

$$\{\det[a_{ij}]\}^2 = 1, \tag{1.19}$$

where $\det[a_{ij}]$ denotes the determinant of a_{ij}. A detailed discussion of determinants is given in section 10 of this chapter. The negative root of equation (1.19) is not considered unless the transformation of axes involves a change of orientation since, for the identity transformation $x_i = x'_i$, $a_{ik} = \delta_{ik}$ and $\det[\delta_{ik}] = 1$. Consequently, $\det[a_{ik}] = 1$, provided the transformations involve only right-handed systems (or left-handed systems).

The transformations (1.8), (1.10), and (1.15) subject to equation (1.17) or (1.18) are known as orthogonal transformations. Three quantities u_i are the components of a vector if, under orthogonal transformation, they transform according to equation (1.15). This may be taken as a definition of a vector. According to this definition, equations (1.8) and (1.10) imply that the representation $\boldsymbol{x}$ of a point is a bound vector since its origin coincides with the origin of the coordinate system.

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If the transformation rule (1.10) holds for coordinate transformations from right-handed systems to left-handed systems (or vice versa), the vector is known as a polar vector. There are scalars and vectors known as pseudo scalars and pseudo or axial vectors; there have transformation rules that involve a change in sign when the coordinate transformation is from a right-handed system to a left-handed system (or vice versa), that is, when $\det[a_{ij}] = -1$. The transformation rule for a pseudo scalar is

$$\phi' = \det[a_{ij}]\phi, \tag{1.20}$$

and for a pseudo vector

$$u'_i = \det[a_{ij}]a_{ij}u_j. \tag{1.21}$$

A pseudo scalar is not a true scalar if a scalar is defined as a single quantity invariant under all coordinate transformations. An example of a pseudo vector is the vector product $\boldsymbol{u} \times \boldsymbol{v}$ of two polar vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. A discussion of the vector product is given in section 9 of this chapter. The moment of a force about a point and the angular momentum of a particle about a point are pseudo vectors. The scalar product of a polar vector and a pseudo vector is a pseudo scalar; an example is the moment of a force about a line. The distinction between pseudo vectors and scalars and polar vectors and true scalars disappears when only right- (or left-) handed coordinate systems are considered. For the development of continuum mechanics presented in this book, only right-handed systems are used.

EXAMPLE PROBLEM 1.1. Show that a rotation through angle π about an axis in the direction of the unit vector $\boldsymbol{n}$ has the transformation matrix

$$a_{ij} = -\delta_{ij} + 2n_i n_j, \det[a_{ij}] = 1.$$

SOLUTION. Referring to Figure 1.3, the position vector of point A has components x_i and point B has position vector with components x'_i.

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