CAMBRIDGE STUDIES IN
ADVANCED MATHEMATICS 106

LINEAR OPERATORS AND THEIR SPECTRA

This wide ranging but self-contained account of the spectral theory of non-self-adjoint linear operators is ideal for postgraduate students and researchers, and contains many illustrative examples and exercises.

Fredholm theory, Hilbert-Schmidt and trace class operators are discussed, as are one-parameter semigroups and perturbations of their generators. Two chapters are devoted to using these tools to analyze Markov semigroups.

The text also provides a thorough account of the new theory of pseudospectra, and presents the recent analysis by the author and Barry Simon of the form of the pseudospectra at the boundary of the numerical range. This was a key ingredient in the determination of properties of the zeros of certain orthogonal polynomials on the unit circle.

Finally, two methods, both very recent, for obtaining bounds on the eigenvalues of non-self-adjoint Schrödinger operators are described. The text concludes with a description of the surprising spectral properties of the non-self-adjoint harmonic oscillator.

CAMBRIDGE STUDIES IN ADVANCED MATHEMATICS

All the titles listed below can be obtained from good booksellers or from Cambridge University Press. For a complete series listing visit:
http://www.cambridge.org/series/sSeries.asp?code=CSAM

Already published

58 J. McCleary *A user's guide to spectral sequences II*
59 P. Taylor *Practical foundations of mathematics*
60 M. P. Brodmann & R. Y. Sharp *Local cohomology*
61 J. D. Dixon *et al. Analytic pro-P groups*
62 R. Stanley *Enumerative combinatorics II*
63 R. M. Dudley *Uniform central limit theorems*
64 J. Jost & X. Li-Jost *Calculus of variations*
65 A. J. Berrick & M. E. Keating *An introduction to rings and modules*
66 S. Morosawa *Holomorphic dynamics*
67 A. J. Berrick & M. E. Keating *Categories and modules with K-theory in view*
68 K. Sato *Levy processes and infinitely divisible distributions*
69 H. Hida *Modular forms and Galois cohomology*
70 R. Iorio & V. Iorio *Fourier analysis and partial differential equations*
71 R. Blei *Analysis in integer and fractional dimensions*
72 F. Borceaux & G. Janelidze *Galois theories*
73 B. Bollobás *Random graphs*
74 R. M. Dudley *Real analysis and probability*
75 T. Sheil-Small *Complex polynomials*
76 C. Voisin *Hodge theory and complex algebraic geometry, I*
77 C. Voisin *Hodge theory and complex algebraic geometry, II*
78 V. Paulsen *Completely bounded maps and operator algebras*
79 F. Gesztesy & H. Holden *Soliton equations and their algebro-geometric solutions, I*
81 S. Mukai *An Introduction to invariants and moduli*
82 G. Tourlakis *Lectures in logic and set theory, I*
83 G. Tourlakis *Lectures in logic and set theory, II*
84 R. A. Bailey *Association schemes*
85 J. Carlson, S. Müller-Stach & C. Peters *Period mappings and period domains*
86 J. J. Duistermaat & J. A. C. Kolk *Multidimensional real analysis I*
87 J. J. Duistermaat & J. A. C. Kolk *Multidimensional real analysis II*
89 M. Golumbic & A. Trenk *Tolerance graphs*
90 L. Harper *Global methods for combinatorial isoperimetric problems*
91 I. Moerdijk & J. Mrcun *Introduction to foliations and Lie groupoids*
92 J. Kollar, K. E. Smith & A. Corti *Rational and nearly rational varieties*
93 D. Applebaum *Levy processes and stochastic calculus*
94 B. Conrad *Modular forms and the Ramanujan conjecture*
95 M. Schecter *An introduction to nonlinear analysis*
96 R. Carter *Lie algebras of finite and affine type*
97 H. L. Montgomery, R. C. Vaughan & M. Schechter *Multiplicative number theory I*
98 I. Chavel *Riemannian geometry*
99 D. Goldfeld *Automorphic forms and L-functions for the group GL(n,R)*
100 M. Marcus & J. Rosen *Markov processes, Gaussian processes, and local times*
101 P. Gille & T. Szamuely *Central simple algebras and Galois cohomology*
102 J. Bertoin *Random fragmentation and coagulation processes*
104 A. Ambrosetti & A. Malchiodi *Nonlinear analysis and semilinear elliptic problems*
105 T. Tao & V. H. Vu *Additive combinatorics*

LINEAR OPERATORS AND THEIR SPECTRA

E. BRIAN DAVIES

CAMBRIDGE UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521866293

First published 2007

Printed in the United Kingdom at the University Press, Cambridge

A catalog record for this publication is available from the British Library

ISBN-13 978-0-521-86629-3 hardback

Contents

Preface

This volume is halfway between being a textbook and a monograph. It describes a wide variety of ideas, some classical and others at the cutting edge of current research. Because it is directed at graduate students and young researchers, it often provides the simplest version of a theorem rather than the deepest one. It contains a variety of examples and problems that might be used in lecture courses on the subject.

It is frequently said that over the last few decades there has been a decisive shift in mathematics from the linear to the non-linear. Even if this is the case it is easy to justify writing a book on the theory of *linear* operators. The range of applications of the subject continues to grow rapidly, and young researchers need to have an accessible account of its main lines of development, together with references to further sources for more detailed reading.

Probability theory and quantum theory are two absolutely fundamental fields of science. In terms of their technological impact they have been far more important than Einstein's relativity theory. Both are entirely linear. In the first case this is in the nature of the subject. Many sustained attempts have been made to introduce non-linearities into quantum theory, but none has yet been successful, while the linear theory has gone from triumph to triumph. Nobody can predict what the future will hold, but it seems likely that quantum theory will be used for a long time yet, even if a non-linear successor is found.

The fundamental equations of quantum mechanics involve self-adjoint and unitary operators. However, once one comes to applications, the situation changes. Non-self-adjoint operators play an important role in topics as diverse as the optical model of nuclear scattering, the analysis of resonances using complex scaling, the behaviour of unstable lasers and the scattering of atoms by periodic electric fields.[1]

[1] A short list of references to such problems may be found in [Berry, website].

There are many routes into the theory of non-linear partial differential equations. One approach depends in a fundamental way on perturbing linear equations. Another idea is to use comparison theorems to show that certain non-linear equations retain desired properties of linear cousins. In the case of the Kortweg-de Vries equation, the exact solution of a highly non-linear equation depends on reducing it to a linear inverse problem. In all these cases progress depends upon a deep technical knowledge of what is, and is not, possible in the linear theory. A standard technique for studying the non-linear stochastic Navier-Stokes equation involves reformulating it as a Markov process acting on an infinite-dimensional configuration space X. This process is closely associated with a *linear* Markov semigroup acting on a space of observables, i.e. bounded functions $f : X \to \mathbf{C}$. The decay properties of this semigroup give valuable information about the behaviour of the original non-linear equation. The material in Section 13.6 is related to this issue.

There is a vast number of applications of spectral theory to problems in engineering, and I mention just three. The unexpected oscillations of the London Millennium Bridge when it opened in 2000 were due to inadequate eigenvalue analysis. There is a considerable literature analyzing the characteristic timbres of musical instruments in terms of the complex eigenvalues of the differential equations that govern their vibrations. Of more practical importance are resonances in turbines, which can destroy them if not taken seriously.

As a final example of the importance of spectral theory I select the work of Babenko, Mayer and others on the Gauss-Kuzmin theorem about the distribution of continued fractions, which has many connections with modular curves and other topics; see [Manin and Marcolli 2002]. This profound work involves many different ideas, but a theorem about the dominant eigenvalue of a certain compact operator having an invariant closed cone is at the centre of the theory. This theorem is close to ideas in Chapter 13, and in particular to Theorem 13.1.9.

Once one has decided to study linear operators, a fundamental choice needs to be made. Self-adjoint operators on Hilbert spaces have an extremely detailed theory, and are of great importance for many applications. We have carefully avoided trying to compete with the many books on this subject and have concentrated on the non-self-adjoint theory. This is much more diverse – indeed it can hardly be called a theory. Studying non-self-adjoint operators is like being a vet rather than a doctor: one has to acquire a much wider range of knowledge, and to accept that one cannot expect to have as high a rate of success when confronted with particular cases. It comprises a collection of methods, each of which is useful for some class of such operators. Some of these are described in the recent monograph of Trefethen and Embree on pseudospectra, Haase's monograph based on the holomorphic functional calculus, Ouhabaz's detailed theory of the L^p semigroups associated with NSA second order elliptic operators, and the much older work of Sz.-Nagy and Foias, still being actively developed by Naboko and others. If there is a common thread in all of these it is the idea of using theorems from analytic function theory to understand NSA operators.

One of the few methods with some degree of general application is the theory of one-parameter semigroups. Many of the older monographs on this subject (particularly my own) make rather little reference to the wide range of applications of the subject. In this book I have presented a much larger number of examples and problems here in order to demonstrate the value of the general theory. I have also tried to make it more user-friendly by including motivating comments.

The present book has a slight philosophical bias towards explicit bounds and away from abstract existence theorems. I have not gone so far as to insist that every result should be presented in the language of constructive analysis, but I have sometimes chosen more constructive proofs, even when they are less general. Such proofs often provide new insights, but at the very least they may be more useful for numerical analysts than proofs which merely assert the existence of a constant or some other entity.

There are, however, many entirely non-constructive proofs in the book. The fact that the spectrum of a bounded linear operator is always non-empty depends upon Liouville's theorem and a contradiction argument. It does not suggest a procedure for finding even one point in the spectrum. It should therefore come as no surprise that the spectrum can be highly unstable under small perturbations of the operator. The pseudospectra are more stable, and because of that arguably more important for non-self-adjoint operators.

It is particularly hard to give precise historical credit for many theorems in analysis. The most general version of a theorem often emerges several decades after the first one, with a proof which may be completely different from the original one. I have made no attempt to give references to the original literature for results discovered before 1950, and have attached the conventional names to theorems of that era. The books of Dunford and Schwartz should be consulted for more detailed information; see [Dunford and Schwartz 1966, Dunford and Schwartz 1963]. I only assign credit on a systematic basis for results proved since 1980, which is already a quarter of a century ago. I may not even have succeeded in doing that correctly, and hope that those who feel slighted will forgive my failings, and let me know, so that the situation can be rectified on my website and in future editions.

I conclude by thanking the large number of people who have influenced me, particularly in relation to the contents of this book. The most important of these have been Barry Simon and, more recently, Nick Trefethen, to both of whom I owe a great debt. I have also benefited greatly from many discussions with Wolfgang Arendt, Anna Aslanyan, Charles Batty, Albrecht Böttcher, Lyonell Boulton, Ilya Goldsheid, Markus Haase, Evans Harrell, Paul Incani, Boris Khoruzhenko, Michael Levitin, Terry Lyons, Reiner Nagel, Leonid Parnovski, Michael Plum, Yuri Safarov, Eugene Shargorodsky, Stanislav Shkarin, Johannes Sjöstrand, Dan Stroock, John Weir, Hans Zwart, Maciej Zworski and many other good friends and colleagues. Finally I want to record my thanks to my wife Jane, whose practical and moral support over many years has meant so much to me. She has also helped me to remember that there is more to life than proving theorems!