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# AN INTRODUCTION TO CONTACT TOPOLOGY

This text on contact topology is the first comprehensive introduction to the subject, including recent striking applications in geometric and differential topology: Eliashberg's proof of Cerf's theorem $\Gamma_4 = 0$ via the classification of tight contact structures on the 3-sphere, and the Kronheimer–Mrowka proof of Property P for knots via symplectic fillings of contact 3-manifolds.

Starting with the basic differential topology of contact manifolds, all aspects of 3-dimensional contact manifolds are treated in this book. One notable feature is a detailed exposition of Eliashberg's classification of overtwisted contact structures. Later chapters also deal with higher-dimensional contact topology. Here the focus is on contact surgery, but other constructions of contact manifolds are described, such as open books or fibre connected sums.

This book serves both as a self-contained introduction to the subject for advanced graduate students and as a reference for researchers.

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# An Introduction to Contact Topology

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Dedicated to the memory of
Charles B. Thomas (1938–2005)

Zu diesem Allen kommt, daß zu Papier gebrachte Gedanken überhaupt nichts weiter sind, als die Spur eines Fußgängers im Sande: man sieht wohl den Weg, welchen er genommen hat; aber um zu wissen, was er auf dem Wege gesehn, muß man seine eigenen Augen gebrauchen.

Arthur Schopenhauer, *Parerga und Paralipomena*

# Contents

# Preface

> 'We are all familiar with the after-the-fact tone — weary, self-justificatory, aggrieved, apologetic — shared by ship's captains appearing before boards of inquiry to explain how they came to run their vessels aground and by authors composing Forewords.'
>
> John Lanchester,
> *The Debt to Pleasure*

Contact geometry, as a subject in its own right, was born in 1896 in the monumental work of Sophus Lie on *Berührungstransformationen* (contact transformations). Lie traces the pedigree of contact geometric notions back to the work of Christiaan Huygens on geometric optics in the *Traité de la Lumière* of 1690 — or even Apollonius of Perga's *Conica* from the third century BC — and to practically all the famous mathematicians of the eighteenth and nineteenth century.

But as late as 1990, when I began my journey into contact geometry, the field still seemed rather arcane. To the prescience of Charles Thomas I owe the privilege of starting graduate work in an area that was only just beginning to flourish. This, of course, had its drawbacks — there were no texts from which to learn the essentials. Even contact geometry's elder sibling, symplectic geometry — firmly established as the natural language for classical mechanics, and brought into prominence by Gromov's influential 1985 paper on pseudoholomorphic curves — suffered from a similar dearth.

One of the most eloquent of modern panegyrists of contact geometry is Vladimir Arnold, who proclaimed on several occasions since 1989 that 'contact geometry is all geometry'. His *Mathematical Methods of Classical Mechanics*, first published (in Russian) in 1974, was then the only textbook covering the basic notions of contact geometry, albeit in the relative obscurity of an appendix.

A *contact structure* is a maximally non-integrable hyperplane field on an odd-dimensional manifold (Defn. 1.1.3). This sounds like a pretty abstruse object, but, as I hope to convince you in Chapter 1, contact structures appear virtually everywhere in nature.†

There are various ways in which contact manifolds may be regarded as the odd-dimensional analogue of symplectic manifolds. They share the property that there is a *Darboux theorem* providing a local model for such structures. Thus, roughly speaking, there are no *local* invariants, but only *global* phenomena of interest.

The investigation of contact manifolds from a more topological perspective was taken up in the late 1950s by John Gray and Boothby–Wang. Gray proved the stability of contact structures on closed manifolds, i.e. the fact that there are no non-trivial deformations of such structures. This is analogous to the stability of symplectic forms proved by Jürgen Moser, and indeed the 'modern' proof of Gray stability uses the famous 'Moser trick'. Boothby and Wang constructed contact structures on certain principal $S^1$–bundles; these were the first non-trivial examples of contact manifolds.

The classical period‡ of contact topology began with the existence results for contact structures on 3–manifolds due to Jean Martinet and Robert Lutz in the early 1970s. Another landmark in this period was Daniel Bennequin's surprising discovery of exotic contact structures on standard 3–space in 1983. The topological flavour of global contact geometry was affirmed in 1989 by Yasha Eliashberg's classification of what he called *overtwisted* contact structures on 3–manifolds; this seminal result marks the culmination of the classical age.

Eliashberg's proof hinted at the importance of using surfaces in contact 3–manifolds as a tool for understanding the latter's global structure. A contact structure induces on such a surface a singular 1–dimensional foliation, the so-called *characteristic foliation*. Control of this characteristic foliation permits various cut-and-paste constructions. Then, in his 1991 thesis, Emmanuel Giroux introduced the notion of a *convex surface*, that is, a surface transverse to a flow preserving the contact structure. It turns out that the characteristic foliation on a convex surface is, in essence, determined by a set of simple closed curves, the *dividing set*, and therefore can be controlled much more effectively.

It seems fair to say that Giroux's thesis inaugurated the modern era of contact topology. Convex surface theory lies at the heart of many of the recent developments, be it the classification of *tight* (i.e. non-overtwisted)

† Admittedly, my notion of 'nature' suffers from a certain *déformation professionnelle*.
‡ On the correct usage of 'classical' see the footnote on page 338.

contact structures on 3–manifolds, or the classification of various types of knots in contact 3–manifolds.

In 1992, Eliashberg's contact geometric proof of Jean Cerf's celebrated result $\Gamma_4 = 0$ gave the first inkling that contact topology might develop into a powerful tool for tackling pure topological questions. In 2004, this expectation was confirmed in a spectacular manner by Peter Kronheimer and Tom Mrowka's establishing Property P for non-trivial knots. (One consequence of this result is a new proof of the Gordon–Luecke theorem that knots in the 3–sphere are determined by their complement.)

The Kronheimer–Mrowka argument is based in an essential way on a result about 'capping off' symplectic fillings of contact 3–manifolds, proved independently by Eliashberg and John Etnyre. This result, in turn, relies crucially on a correspondence between contact structures and open book decompositions, established by Giroux. This correspondence is another instance of the topological nature of much of contact geometry.

One of the aims of this book is to explain these exciting developments, starting from an exposition of first principles. As yet, there is no textbook or monograph giving a comprehensive introduction to contact geometry. Some basic aspects are treated in Chapter 8 of [7] and in Section 3.4 of [177], but these texts do not prepare the reader for any of the modern topological techniques in contact geometry. The introductory lectures by Etnyre [80], though brief, give a better impression of the state of the art. In the *Handbook* article [97] I was more thorough, but my focus lay on the basic differential topological aspects — many of which had remained folklore until then — and no attempt was made to cover more than the classical period.

Fortunately, that article could serve as a germ for the present monograph and now constitutes part of Chapters 2, 3, and Sections 4.1 to 4.5. Otherwise the task of writing might have seemed too daunting.

Previous versions of Sections 4.1 to 4.3 and Chapter 8 were written as early as 1997 for a series of lectures I gave in Les Diablerets as part of the IIIe Cycle Romand, but contact topology simply developed too rapidly for an exhaustive account to be feasible even then. That a tortoise cannot catch up with Achilles is hardly paradoxical. So when I started writing this book in earnest, I had to be modest and realistic about its aims. The reader I had primarily in mind is an advanced graduate student educated in differential topology and the geometric aspects of algebraic topology, say on the level of the textbooks by Bröcker–Jänich [38] or Hirsch [132], and that by Bredon [35], respectively. A good dose of geometric topology, e.g. from Rolfsen [215] or Prasolov–Sossinsky [209], would be helpful, but most geometric topological concepts will be explained at length when we need them. The present book should serve as a preparation for studying a substantial body of research literature; ample references for further reading will be provided.

Here is a brief summary of the contents of this book; more details can be found at the beginning of each chapter. Chapter 1 is something of an advertisement, aimed at readers not necessarily inclined to devote the rest of their life to contact geometry. Chapter 2 deals with the basic differential topology of contact manifolds. Sections 2.1 to 2.4 are foundational for everything that is to follow. The neighbourhood and isotopy extension theorems in Sections 2.5 and 2.6 are likewise indispensable, but their proofs — though useful for learning the nuts and bolts of the Moser trick — should probably be skimmed on a first reading. The contact disc theorem in Section 2.6.2 deserves special mention. This theorem has been assumed at various places in the contact topological literature. The proof is analogous to that of the usual disc theorem in differential topology, but I thought it opportune to be explicit about the 'contact Alexander trick'.

Chapter 3 provides a basic introduction to the theory of knots in contact 3–manifolds. There are two types of knots to investigate: those tangent to the contact structure (called *Legendrian knots*), and those transverse to the contact structure (called, rather imaginatively, *transverse knots*). That chapter constitutes the basis for the surgical constructions of contact manifolds along transverse knots (Sections 4.1 to 4.3) and Legendrian knots (Section 6.4), respectively.

My initial plan for Chapter 4 on contact 3–manifolds was to heed Polonius's advice that 'brevity is the soul of wit' [220]. If that chapter has grown beyond proportion, it is less owing to the 'outward flourishes', but to my realising that this monograph would be inadequate if it failed to introduce the reader to some modern techniques like convex surfaces (Sections 4.6.2 and 4.8) and Giroux's tomography (Section 4.9). Even so, there are some regrettable omissions. I say virtually nothing about Ko Honda's important contributions to convex surface theory — Honda's approach is parallel to, but independent of, much of Giroux's work on tomography. My impression is that tomography *à la* Giroux is more amenable to an elementary treatment, in particular for the classification of tight contact structures on some simple 3–manifolds (Section 4.10).

Concerning the relation between open books and contact structures, I only discuss the *existence* of contact structures on open books, both in dimension 3 (Section 4.4.2) and in higher dimensions (Section 7.3). For the converse, how to find an open book decomposition adapted to a given contact structure, there are some useful lecture notes by Etnyre [83], at least for the 3–dimensional case.

Other lacunæ in the chapter on 3–manifolds stem from my failure to mention anything related to the Heegaard Floer theory of Peter Ozsváth and Zoltán Szabó, or the analytical side of contact geometry (dynamics of the Reeb flow, including the Weinstein conjecture, and techniques involving pseudoholomorphic curves), initiated by Helmut Hofer. Here are some pertinent references: [165], [200], [201], [202], [226]; [135], [136], [228].

Section 4.7 contains a detailed proof of Eliashberg's classification of overtwisted contact structures, following the lines of the original argument. Eliashberg's paper — together with his 1992 paper on Cerf's theorem — is probably one of the two most-quoted papers in all of contact geometry, but one which is notoriously hard to read. I hope that my exposition goes some way towards making Eliashberg's ideas more accessible.†

Chapter 4 also includes Eliashberg's proof of Cerf's theorem (Section 4.11), and the analogue for tight contact 3–manifolds of John Milnor's unique decomposition theorem for 3–manifolds (Section 4.12, based on joint work with Fan Ding).

The remainder of the book is concerned mostly with higher-dimensional contact topology. Chapter 5 introduces the notions of symplectic fillings and convexity. There is an intrinsic interest in relating contact structures to notions of convexity in complex geometry, but the main purpose of that chapter is to lead into Chapter 6 on contact surgery. Being the mathematical son of a mathematical surgeon and the real grandson of a real surgeon, I regard this chapter (and its companion Chapter 8, where the theory is applied to 5–manifolds) as my earliest motivation for writing this book. Sections 6.1 to 6.3 contain an exposition of the Eliashberg–Weinstein contact surgery. Some ideas about homotopy principles *à la* Gromov are explained along the way. I also allow myself a small detour to give a contact geometric proof of the Whitney–Graustein theorem on the classification of immersions of the circle in the plane.

With Section 6.4 we return to the study of contact 3–manifolds. In the 3–dimensional setting one has a more general concept of contact *Dehn* surgery, introduced by Fan Ding and myself. This type of surgery is then used to prove the Eliashberg–Etnyre theorem about symplectic caps — in contrast with the proofs of Eliashberg and Etnyre, the one presented here does not rely on the Giroux correspondence between contact structures and open books. That section completes the account of the contact topological aspects of the Kronheimer–Mrowka proof of Property P.

In Section 6.5, contact surgery of 3–manifolds is used to show, modulo a result from Seiberg–Witten theory that I quote only, that symplectically fillable contact structures are tight.

† Charles Thomas might have referred to this exposition — with apologies to Moses Maimonides — as a 'guide for the perplexed'.

As mentioned before, Chapter 8 may be regarded as a rather long example how contact surgery can be applied. Section 8.2 is an exercise in the classification of manifolds; Section 8.1 may be instructive as an application of obstruction theory, and it contains some neat geometry of complex projective 3–space.

Chapter 7, which can be read right after Chapter 2, gives further justification — if such be required — for calling this a book on contact *topology*. That chapter explains a variety of topological constructions that can be made compatible with contact structures. Besides the open books referred to earlier, these are fibre connected sums, branched covers, and plumbing. Applications of these techniques include the construction of contact structures on all odd-dimensional tori, due to Frédéric Bourgeois, and that of exotic contact structures on all odd-dimensional spheres. Here I have to confess another glaring sin: contact homology, not to mention symplectic field theory, gets only a passing reference as a tool for distinguishing such exotic structures. Slightly more geometric constructions discussed in Chapter 7 are Brieskorn manifolds, the Boothby–Wang construction, and contact reduction.

The two brief appendices contain some minor technicalities concerning the generalised Poincaré lemma and time-dependent vector fields.

Exercises are conspicuous by their absence. I hope that the numerous explicit examples make up for this defect.

As pointed out earlier, this book had its beginnings in a series of lectures in the IIIe Cycle Romand in 1997. The book began to take shape during a sabbatical at the Forschungsinstitut für Mathematik of the ETH Zürich in 2004. I lectured on parts of this book in various seminars, and in 2005 at a Frühlingsschule in Waren (Müritz) organised by Christian Becker, Hartmut Weiss and Anna Wienhard. In the winter term 2006/07 I gave a lecture course on contact geometry at the Universität zu Köln. In June 2007 I presented the Lisbon Summer Lectures in Geometry based on this book, at the invitation of Miguel Abreu and Sílvia Anjos. The questions and comments of these various audiences have helped to improve the exposition.

Many people read parts of this book and found numerous smaller or bigger misprints, obscure statements, or plain inaccuracies. Countless stimulating conversations have left their traces in this book. For their assistance I thank Frédéric Bourgeois, Yuri Chekanov, Yvonne Deuster, Thomas Eckl, John Etnyre, Otto van Koert, Paolo Lisca, Klaus Niederkrüger, Federica Pasquotto, Bijan Sahamie, Felix Schlenk and Matthias Zessin. Fabian Meier sent me a copy of his Cambridge Part III essay on Eliashberg's classification of overtwisted contact structures; this helped with working out some of the details

in Section 4.7. Stephan Schönenberger allowed me to use his Figures 1.1 and 2.1; John Etnyre contributed Figures 2.2 and 6.8. John Etnyre's encouraging me to go ahead with this book project is invaluable. I am grateful to my collaborators Fan Ding, Jesús Gonzalo and András Stipsicz; talking mathematics with them has been a wonderful experience over many years, and the fruits of our joint projects are visible throughout this book.

Yasha Eliashberg has been a source of inspiration ever since he commented encouragingly on my first mathematical publication. His influence on this book is plainly evident. Especially Section 4.7 would never have been completed without his spending many hours patiently explaining to me his classification of overtwisted contact structures. In passing we noticed little gems such as the contact geometric proof of the Whitney–Graustein theorem.

Much of my intellectual education, mathematical and otherwise, I owe to my mentor and friend Charles Thomas. I should have liked to present this book to him as a small thank-you for introducing me to the fascinating world of contact geometry when I was his graduate student and — perhaps even more important — for showing me by his example what it means to lead a scholarly life. Now, sadly, all I can do is dedicate this book to his memory.

Köln, July 2007 Hansjörg Geiges