1
Facets of contact geometry

> 'After a while the style settles down a bit and it begins to tell you things you really need to know.'
>
> Douglas Adams,
> *The Hitch Hiker's Guide to the Galaxy*

This opening chapter is not meant as an introduction in the conventional sense (if only because it is much too long for that). Perhaps it would be appropriate to call it a *proem*, defined by the *Oxford English Dictionary* as 'an introductory discourse at the beginning of a book'. Although some basic concepts are introduced along the way, the later chapters are largely independent of the present one. Primarily this chapter gives a somewhat rambling tour of contact geometry.

Specifically, we consider polarities in projective geometry, the Hamiltonian flow of a mechanical system, the geodesic flow of a Riemannian manifold, and Huygens' principle in geometric optics. Contact geometry is the theme that connects these diverse topics. This may serve to indicate that Arnold's claim that 'contact geometry is all geometry' ([15], [17]) is not entirely facetious. I also present two remarkable applications of contact geometry to questions in differential and geometric topology: Eliashberg's proof of Cerf's theorem $\Gamma_4 = 0$ via the classification of contact structures on the 3–sphere, and the proof of Property P for non-trivial knots by Kronheimer and Mrowka, where symplectic fillings of contact manifolds play a key role. One of the major objectives of this book will be to develop the contact topological methods necessary to understand these results.

The subsequent chapters will not rely substantively on the material presented here. Therefore, readers interested in a more formal treatment of contact geometry and topology may skim this proem and refer back to it for some basic definitions only. But I hope that the present chapter, while serving as an invitation to contact geometry for the novice, contains material that will be of interest even to readers with prior exposure to the subject. In the survey article [94] I have touched upon some of the themes of this introduction, but without the detailed proofs given here; on the other hand, that article contains more on the historical development of contact geometry. On that topic, see also the historical survey by Lutz [170]. As an *amuse-gueule* you may enjoy [96].

1.1 Contact structures and Reeb vector fields

Let M be a differential manifold, TM its tangent bundle, and $\xi \subset TM$ a field of hyperplanes on M, that is, a smooth† sub-bundle of codimension 1. The term codimension 1 **distribution** is quite common for such a tangent hyperplane field (and not to be confused with distributions in the analysts' sense, of course). In order to describe special types of hyperplane fields, it is useful to present them as the kernel of a differential 1–form.

Lemma 1.1.1 *Locally, ξ can be written as the kernel of a differential 1–form α. It is possible to write $\xi = \ker \alpha$ with a 1–form α defined globally on all of M if and only if ξ is* coorientable*, which by definition means that the quotient line bundle TM/ξ is trivial.*

Proof Choose an auxiliary Riemannian metric g on M and define the line bundle $\xi^\perp$ as the orthogonal complement of ξ in TM with respect to that metric. Then $TM \cong \xi \oplus \xi^\perp$ and $TM/\xi \cong \xi^\perp$. Around any given point p of M, there is a neighbourhood $U = U_p$ over which the line bundle $\xi^\perp$ is trivial. Let X be a non-zero section of $\xi^\perp|_U$ and define a 1–form α_U on U by $\alpha_U = g(X, -)$. Then clearly $\xi|_U = \ker \alpha_U$.

Saying that ξ is coorientable is the same as saying that $\xi^\perp$ is orientable and hence (being a line bundle) trivial. In that case, X and thus also α exist globally. Conversely, if $\xi = \ker \alpha$ with a globally defined 1–form α, one can define a global section of $\xi^\perp$ by the conditions $g(X, X) \equiv 1$ and $\alpha(X) > 0$, hence ξ is coorientable. □

Remark 1.1.2 As a student at Cambridge, I was taught that in Linear Algebra 'a gentleman should never use a basis unless he really has to'. By the same token, one should not use auxiliary Riemannian metrics if one can do without them. So here is the alternative argument, which may well serve

† The terms *smooth* and *differentiable* are used synonymously with C^∞; a *differential* manifold is a manifold with a choice of differentiable structure. Throughout this book, manifolds, bundles, vector fields, and related objects, are assumed to be smooth.

as a warm-up for (slightly less pedestrian) bundle-theoretic considerations later on.

The quotient bundle TM/ξ and its dual bundle are locally trivial. Thus, over small neighbourhoods U, one can define a differential 1–form α_U as the pull-back to the cotangent bundle T^*M of a non-zero section of $(TM/\xi)^*|_U$ under the bundle projection $TM \to TM/\xi$. This 1–form clearly satisfies $\ker \alpha_U = \xi|_U$.

If ξ is coorientable, $(TM/\xi)^*$ admits a global section, and the above construction yields a global 1–form α defining ξ. Conversely, a global 1–form α defining $\xi = \ker \alpha$ induces a global non-zero section of $(TM/\xi)^*$.

Except in certain isolated examples below, we shall always assume our hyperplane fields ξ to be coorientable.

One class of hyperplane fields that has received a great deal of attention are the **integrable** ones. This term denotes hyperplane fields with the property that through any point $p \in M$ one can find a codimension 1 submanifold N whose tangent spaces coincide with the hyperplane field, i.e. such that $T_qN = \xi_q$ for all $q \in N$. Such an N is called an **integral submanifold** of ξ. It turns out that $\xi = \ker \alpha$ is integrable precisely if α satisfies the *Frobenius integrability condition*

$$\alpha \wedge d\alpha \equiv 0.$$

In terms of Lie brackets of vector fields, this integrability condition can be written as

$$[X, Y] \in \xi \text{ for all } X, Y \in \xi;$$

here $X \in \xi$ means that X is a smooth section of TM with $X_p \in \xi_p$ for all $p \in M$. A third equivalent formulation of integrability is that ξ is locally of the form $dz = 0$, where z is a local coordinate function on M. A good textbook reference for these facts is Warner [238]. The collection of integral submanifolds of an integrable hyperplane field constitutes what is called a codimension 1 *foliation*. For the global topology of foliations (of arbitrary codimension) see Tamura [227].

Contact structures are in a certain sense the exact opposite of integrable hyperplane fields.

Definition 1.1.3 Let M be a manifold of odd dimension $2n+1$. A **contact structure** is a maximally non-integrable hyperplane field $\xi = \ker \alpha \subset TM$, that is, the defining differential 1–form α is required to satisfy

$$\alpha \wedge (d\alpha)^n \neq 0$$

(meaning that it vanishes nowhere). Such a 1–form α is called a **contact form**. The pair (M, ξ) is called a **contact manifold**.

Remark 1.1.4 As a somewhat degenerate case, this definition includes 1–dimensional manifolds with a non-vanishing 1–form α. The corresponding contact structure $\xi = \ker \alpha$ is the zero section of the tangent bundle.

Example 1.1.5 On $\mathbb{R}^{2n+1}$ with Cartesian coordinates

$$(x_1, y_1, \ldots, x_n, y_n, z),$$

the 1–form

$$\alpha_1 = dz + \sum_{j=1}^{n} x_j \, dy_j$$

is a contact form. The contact structure $\xi_1 = \ker \alpha_1$ is called the **standard contact structure** on $\mathbb{R}^{2n+1}$. See Figure 1.1 for the 3–dimensional case.

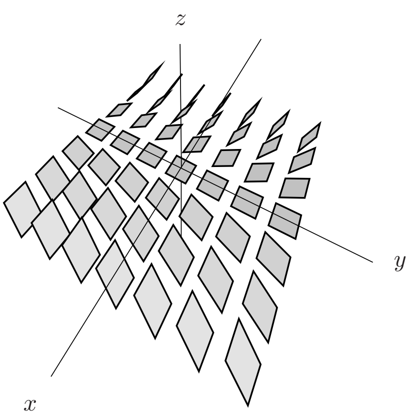

Fig. 1.1. The contact structure $\ker(dz + x\,dy)$.

Remark 1.1.6 Observe that α is a contact form precisely if $\alpha \wedge (d\alpha)^n$ is a **volume form** on M (i.e. a nowhere vanishing top-dimensional differential form); in particular, M needs to be orientable. The condition $\alpha \wedge (d\alpha)^n \neq 0$ is independent of the specific choice of α and thus is indeed a property of $\xi = \ker \alpha$: any other 1–form defining the same hyperplane field must be of the form $\lambda\alpha$ for some smooth function $\lambda \colon M \to \mathbb{R} \setminus \{0\}$, and we have

$$(\lambda\alpha) \wedge (d(\lambda\alpha))^n = \lambda\alpha \wedge (\lambda \, d\alpha + d\lambda \wedge \alpha)^n = \lambda^{n+1} \alpha \wedge (d\alpha)^n \neq 0.$$

We see that if n is odd, the sign of this volume form depends only on ξ, not the choice of α, so the contact structure ξ induces a natural orientation of M. If M comes equipped with a specific orientation, one can speak of *positive* and *negative* contact structures.

Lemma 1.1.7 *In the 3–dimensional case the contact condition can also be formulated as*

$$[X,Y]_p \notin \xi_p \quad \text{at every } p \in M, \text{ for all pointwise linearly independent vector fields } X, Y \in \xi.$$

Proof The equation

$$d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]),$$

which holds for arbitrary 1–forms α and vector fields X, Y on M, see [238, p. 70], implies

$$d\alpha(X,Y) = -\alpha([X,Y]) \text{ for all } X, Y \in \xi = \ker \alpha.$$

The contact condition $\alpha \wedge d\alpha \neq 0$ in dimension 3 is equivalent to $d\alpha|_\xi \neq 0$. This implies the claim. □

Example 1.1.8 The standard contact structure ξ_1 on $\mathbb{R}^3$ is given by $dz + x\,dy = 0$, hence ξ_1 is spanned by the vector fields ∂_x and $\partial_y - x\,\partial_z$, with $[\partial_x, \partial_y - x\,\partial_z] = -\partial_z \notin \xi_1$.

Here is another fundamental concept of contact geometry.

Lemma/Definition 1.1.9 *Associated with a contact form α one has the so-called* **Reeb vector field** R_α, *uniquely defined by the equations*

(i) $d\alpha(R_\alpha, -) \equiv 0$,
(ii) $\alpha(R_\alpha) \equiv 1$.

Proof This is essentially a matter of linear algebra. For each point $p \in M$, the form $d\alpha|_{T_pM}$ is, by the contact condition $\alpha \wedge (d\alpha)^n \neq 0$, a skew-symmetric form of maximal rank $2n$ (for M of dimension $2n+1$). Hence $d\alpha|_{T_pM}$ has a 1–dimensional kernel (see the section on symplectic linear algebra below) and equation (i) defines R_α uniquely up to scaling, in other words, a unique line field $\langle R_\alpha \rangle \subset TM$. (The smoothness of this line field follows from the smoothness of α.) Again by the contact condition, α is non-trivial on that line field, so the normalisation condition (ii) specifies a non-vanishing section of it. □

Example 1.1.10 The Reeb vector field R_{α_1} of the standard contact form α_1 on $\mathbb{R}^{2n+1}$ equals ∂_z.

Notice that one cannot reasonably speak of the Reeb vector field of a contact *structure*. The consequences of the fact that different contact forms defining the same contact structure may have Reeb vector fields with wildly differing dynamics will be addressed in Example 2.2.5 below.

The name 'contact structure' has its origins in the fact that one of the first historical sources of contact manifolds are the so-called *spaces of contact elements*. We shall presently discuss these and other classical examples of contact manifolds.

1.2 The space of contact elements

In 1872, Lie [159] (see also [160], [161]) introduced the notion of *contact transformation* (Berührungstransformation) as a geometric tool for studying systems of differential equations. This may be regarded as the earliest precursor of modern contact geometry.

Contact transformations constitute a particular case of a local transformation group defined by the integrals of a system of differential equations. These transformations were studied extensively during the later part of the nineteenth century and the beginning of the twentieth century by, amongst others, Engel, Poincaré, Goursat, and Cartan.

In the present section we phrase in modern language some of the contact geometric notions that can be traced back to the work of Lie.

Definition 1.2.1 Let B be a smooth n–dimensional manifold. A **contact element** is a hyperplane in a tangent space to B. The **space of contact elements** of B is the collection of pairs (b, V) consisting of a point $b \in B$ and a contact element $V \subset T_b B$.

Lemma 1.2.2 *The space of contact elements of B can be naturally identified with the projectivised cotangent bundle $\mathbb{P}T^*B$, which is a manifold of dimension $2n-1$.*

Proof A hyperplane V in the tangent space $T_b B$ is defined as the kernel of a non-trivial linear map $u_V : T_b B \to \mathbb{R}$, and u_V is determined by V up to multiplication by a non-zero scalar. So the space of contact elements at $b \in B$ may be thought of as the projectivisation of the dual space $T_b^* B$. It is standard bundle theory that this fibrewise projectivisation yields a smooth bundle, see [38]. □

Next we want to see that the space of contact elements comes equipped with a contact structure.

Lemma 1.2.3 *Write π for the bundle projection $\mathbb{P}T^*B \to B$. For $u = u_V \in \mathbb{P}T_b^*B$, let ξ_u be the hyperplane in $T_u(\mathbb{P}T^*B)$ such that $T\pi(\xi_u)$ is the hyperplane V in $T_{\pi(u)}B = T_bB$ defined by u. Then ξ defines a contact structure on $\mathbb{P}T^*B$.*

We call this the **natural contact structure** on the space of contact elements. Figure 1.2 illustrates the construction for $B = \mathbb{R}^2$. Here $\mathbb{P}T^*B = \mathbb{R}^2 \times \mathbb{R}P^1$.

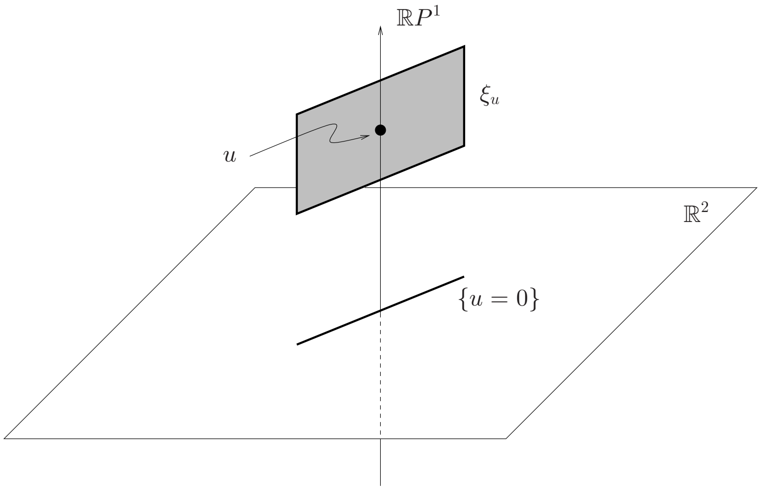

Fig. 1.2. The space of contact elements.

Proof of Lemma 1.2.3 Let $q_1, \ldots, q_n$ be local coordinates on B, and denote the corresponding dual coordinates in the fibres of the cotangent bundle T^*B by $p_1, \ldots, p_n$. This means that the coordinate description of covectors is given by

$$(q_1, \ldots, q_n, p_1, \ldots, p_n) = \left(\sum_{j=1}^{n} p_j \, dq_j \right)_{(q_1, \ldots, q_n)}.$$

Thus, a point

$$(q_1, \ldots, q_n, (p_1 : \ldots : p_n))$$

in the projectivised cotangent bundle $\mathbb{P}T^*B$ defines the hyperplane

$$\sum_{j=1}^{n} p_j \, dq_j = 0$$

in T_bB, where $b = (q_1, \ldots, q_n)$. By construction, the natural contact structure ξ on $\mathbb{P}T^*B$ is defined by

$$\xi = \ker\Big(\sum_{j=1}^{n} p_j \, dq_j\Big);$$

notice that this kernel is indeed well defined in terms of the coordinates on $\mathbb{P}T^*B$, although the 1–form $\sum p_j \, dq_j$ is not.

In order to verify the contact condition for ξ, we restrict to affine subspaces of the fibre. Over the open set $\{p_1 \neq 0\}$,† for instance, ξ is defined in terms of affine coordinates $p'_j = p_j/p_1$, $j = 2, \ldots, n$, by the equation

$$dq_1 + p'_2 \, dq_2 + \cdots + p'_n \, dq_n = 0,$$

which is exactly the description of the standard contact structure on $\mathbb{R}^{2n-1}$ from Example 1.1.5. □

Example 1.2.4 Consider the 2–torus $B = T^2 = S^1 \times S^1$ and let x, y be S^1–valued coordinates on B. Since B has trivial (co-)tangent bundle, the space of contact elements of B is $B \times \mathbb{R}P^1$. When identifying $\mathbb{R}P^1$ with $\mathbb{R}/\pi\mathbb{Z}$ with coordinate θ, the natural contact structure can be written as

$$\sin\theta \, dx - \cos\theta \, dy = 0.$$

This is an example of a contact structure that is not coorientable. It lifts to a coorientable contact structure, given by the same equation, on $B \times S^1$, with $S^1 := \mathbb{R}/2\pi\mathbb{Z}$.

This standard contact structure on the space of contact elements also plays a role in the Hamiltonian formalism of classical mechanics, a point to which we shall return in Section 1.4.

Definition 1.2.5 A **contact transformation** is a diffeomorphism of a space of contact elements that preserves the natural contact structure on that space.

† By this notation I mean the set of points (here: in the projectivised cotangent bundle over a coordinate neighbourhood) that satisfy the inequality in braces. Similar shorthand notation will be used throughout this text.

Cambridge University Press
978-0-521-86585-2 - An Introduction to Contact Topology
Hansjorg Geiges
Excerpt
More information

We briefly elaborate on the relevance of contact transformations for the theory of differential equations. Let $B = \mathbb{R}^2$. In this case, contact elements are also called **line elements**. Following Lie, we write (x, z) for the Cartesian coordinates on $\mathbb{R}^2$, and p for the slope of a line passing through a given point. The space of line elements whose slope p is finite can be identified with $\mathbb{R}^3$ with coordinates (x, z, p). The equation for lines of slope p is given by

$$dz - p\,dx = 0,$$

and when regarded as an equation on the space of line elements, it defines the natural contact structure. A solution $z = z(x)$ of a differential equation $F(x, z, z') = 0$ corresponds to an integral curve

$$x \longmapsto (x, z(x), z'(x))$$

of that contact structure.

Observe that a diffeomorphism

$$f\colon\ (x, z, p) \longmapsto (x_1, z_1, p_1)$$

of $\mathbb{R}^3$ is a contact transformation if and only if

$$dz_1 - p_1\,dx_1 = \rho(dz - p\,dx)$$

for some nowhere zero function $\rho\colon \mathbb{R}^3 \to \mathbb{R}$, that is,

$$f^*(dz - p\,dx) = \rho(dz - p\,dx).$$

Equivalently, this is saying that f maps all integral curves of the contact structure $dz - p\,dx = 0$ to integral curves of $dz_1 - p_1\,dx_1 = 0$.

Define a function F_1 of the transformed coordinates (x_1, z_1, p_1) by

$$F_1(x_1, z_1, p_1) = F(x, z, p),$$

i.e. $F = F_1 \circ f$.

Lemma 1.2.6 *Let f be a contact transformation as above. Suppose the curve $x \mapsto z(x)$ is a local solution of the differential equation*

$$F(x, z, z') = 0.$$

Define a curve in the transformed variables by

$$(x_1(x), z_1(x), p_1(x)) := f(x, z(x), z'(x)).$$

If the curve $x \mapsto (x_1(x), z_1(x))$ is regular, i.e. $(x_1'(x), z_1'(x)) \neq (0,0)$ for all x, *then $z_1(x)$ may be regarded as a function of the transformed variable $x_1(x)$, and the curve $x_1 \mapsto z_1(x_1)$ is a local solution of the transformed equation*

$$F_1\left(x_1, z_1, \frac{dz_1}{dx_1}\right) = 0.$$

Proof The curve $x \mapsto (x, z(x), z'(x))$ is an integral curve of the natural contact structure $dz - p\,dx = 0$. Since f is a contact transformation, the curve

$$x \longmapsto (x_1(x), z_1(x), p_1(x)) := f(x, z(x), z'(x))$$

is an integral curve of the contact structure $dz_1 - p_1\,dx_1 = 0$. It follows that $z_1'(x) - p_1(x)x_1'(x) = 0$. With the regularity condition on the curve $x \mapsto (x_1(x), z_1(x))$ this forces $x_1'(x) \neq 0$. We may therefore write x locally as a function of x_1. Hence

$$\frac{dz_1}{dx_1}(x_1) = \frac{dz_1}{dx}(x(x_1)) \cdot \frac{dx}{dx_1}(x_1) = \frac{z_1'(x(x_1))}{x_1'(x(x_1))} = p_1(x(x_1))$$

and

$$\begin{aligned} F_1\left(x_1, z_1, \frac{dz_1}{dx_1}\right) &= F_1(x_1(x), z_1(x), p_1(x)) \\ &= F_1 \circ f(x, z(x), z'(x)) \\ &= F(x, z(x), z'(x)) = 0. \end{aligned}$$

This proves the lemma. □

Example 1.2.7 Let $z\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto z(x)$ be a strictly convex function (i.e. $z'' > 0$), and assume that $x \mapsto z'(x)$ defines a diffeomorphism of the real line onto itself. Then, for any $p \in \mathbb{R}$, the function $Z_p\colon \mathbb{R} \to \mathbb{R}$ defined by

$$Z_p(x) := px - z(x)$$

has a unique maximum at the point $x = x(p)$ given by

$$\frac{dz}{dx}(x(p)) = p.$$

Define a new function z_1 of the variable p by

$$z_1(p) = Z_p(x(p)) = p \cdot x(p) - z(x(p)).$$

Then

$$\frac{dz_1}{dp}(p) = x(p) + p \cdot \frac{dx}{dp}(p) - \frac{dz}{dx}(x(p)) \cdot \frac{dx}{dp}(p) = x(p).$$

The transformation

$$f\colon (x, z, p) \longmapsto (x_1 := p, z_1 := px - z, p_1 := x)$$