1 The Lagrange Equations of Motion

1.1 Introduction

A knowledge of the rudiments of dynamics is essential to understanding structural dynamics. Thus this chapter reviews the basic theorems of dynamics without any consideration of structural behavior. This chapter is preliminary to the study of structural dynamics because these basic theorems cover the dynamics of both rigid bodies and deformable bodies. The scope of this chapter is quite limited in that it develops only those equations of dynamics, summarized in Section 1.10, that are needed in subsequent chapters for the study of the dynamic behavior of (mostly) elastic structures. Therefore it is suggested that this chapter need only be read, skimmed, or consulted as is necessary for the reader to learn, review, or check on (i) the fundamental equations of rigid/flexible body dynamics and, more importantly, (ii) to obtain a familiarity with the Lagrange equations of motion.

The first part of this chapter uses a vector approach to describe the motions of masses. The vector approach arises from the statement of Newton's second and third laws of motion, which are the starting point for all the material in this textbook. These vector equations of motion are used only to prepare the way for the development of the scalar Lagrange equations of motion in the second part of this chapter. The Lagrange equations of motion are essentially a reformulation of Newton's second law in terms of work and energy (stored work). As such, the Lagrange equations have the following three important advantages relative to the vector statement of Newton's second law: (i) the Lagrange equations are written mostly in terms of point functions that sometimes allow significant simplification of the geometry of the system motion, (ii) the Lagrange equations do not normally involve either external or internal reaction forces and moments, and (iii) the Lagrange equations have the same mathematical form regardless of the choice of the coordinates used to describe the motion. These three advantages alone are sufficient reasons to use the Lagrange equations throughout the remaining chapters of this textbook.

1.2 Newton's Laws of Motion

Newton's three laws of motion can be paraphrased as (Ref. [1.1]):

1. Every particle continues in its state of rest or in its state of uniform motion in a straight line unless it is compelled to change that state by forces impressed upon it.

2. The time rate of change of momentum is proportional to the impressed force, and it is in the direction in which the force acts.

3. Every action is always opposed by an equal reaction.

These three laws are not the only possible logical starting point for the study of the dynamics of masses. However, (i) these three laws are at least as logically convenient as any other complete basis for the motion of masses, (ii) historically, they were the starting point for the development of the topic of the dynamics, and (iii) they are the one basis that almost all readers will have in common. Therefore they are the starting point for the study of dynamics in this textbook.

There are features of this statement of Newton's laws that are not immediately evident. The first of these is that these laws of motion are stated for a single particle, which is a body of very, very small spatial dimensions, but with a fixed, finite mass. The mass of the jth particle is symbolized as m_j. The second thing to note is that momentum, which means rectilinear momentum, is the product of the mass of the particle and its instantaneous velocity. Of course, mass is a scalar quantity, whereas velocity and force are vector quantities. Hence the second law is a vector equation. The third thing to note is that the second law, which includes the first law, is not true for all coordinate systems. The best that can be said is that there is a *Cartesian* coordinate system "in space" for which the second law is valid. Then it is easy to prove (see the first exercise) that the second law is also true for any other Cartesian coordinate system that translates at a constant velocity relative to the valid coordinate system. The second law is generally not true for a Cartesian coordinate system that rotates relative to the valid coordinate system. However, as a practical matter, it is satisfactory to use a Cartesian coordinate system fixed to the Earth's surface *if* the duration of the motion being studied is only a matter of a few minutes. The explanation for this exception is that the rotation of the Cartesian coordinate system fixed at a point on the Earth's surface at the constant rate of one-quarter of a degree per minute, or 0.0007 rpm, mostly just translates that coordinate system at the earth's surface in that short period of time. See Figure 1.1(a).

As is derived below, when Newton's second law is extended to a mass m of finite spatial dimensions, which is subjected to a net external force of magnitude[1] $\boldsymbol{F}$, then Newton's second law can be written in vector form as follows:

$$\boldsymbol{F} = \frac{d\boldsymbol{P}}{dt} = m\frac{d\boldsymbol{v}}{dt} = m\boldsymbol{a}, \tag{1.1}$$

where $\boldsymbol{P} = m\boldsymbol{v}$ is the momentum vector, $\boldsymbol{v}$ is the velocity vector of the total mass m relative to the valid coordinate origin, t is time, and $\boldsymbol{a}$ is the acceleration vector, which of course is the time derivative of the velocity vector. The velocity vector is not the

[1] Vector quantities are indicated by the use of italic boldface type.

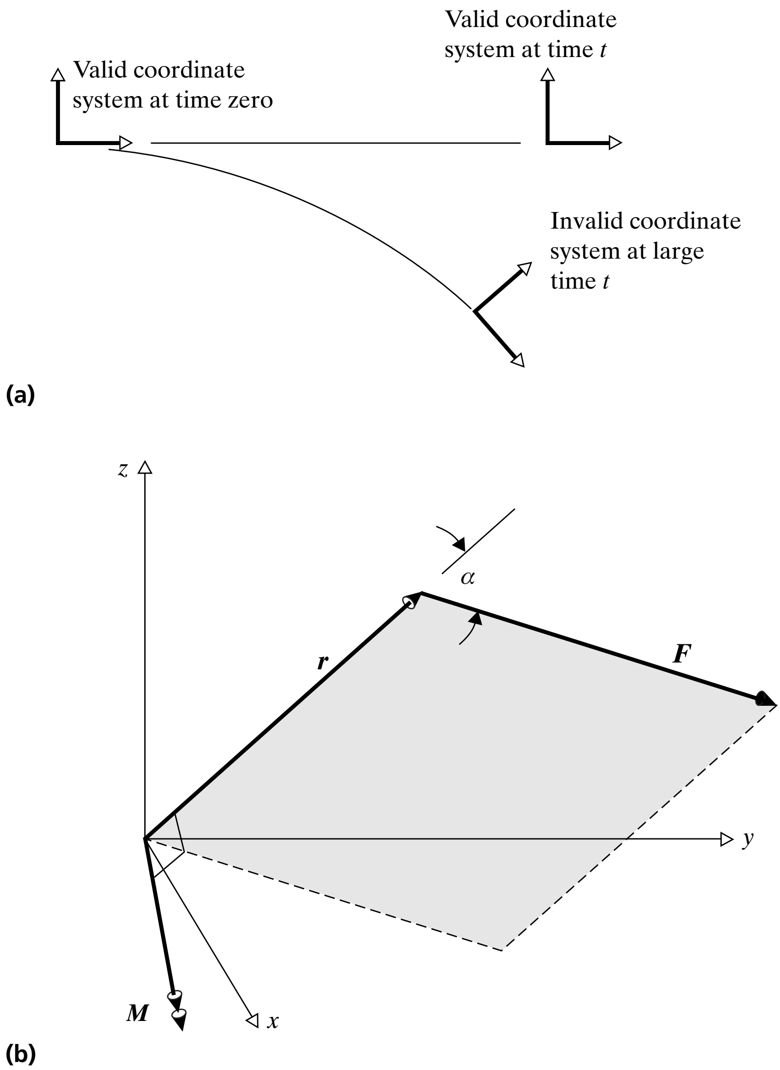

Figure 1.1. (a) Valid and invalid coordinate systems for Newton's second law, both moving at constant speed. (b) Illustration of the right-hand rule for $\boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{M} = rF \sin \alpha \boldsymbol{n}$.

velocity of all points within the mass m relative to the valid coordinate system. Rather, it is the velocity of the one point called the *center of mass*, which is defined below. Further, note that the mass of the system of particles whose motion is described by this equation is the mass of a fixed collection of specific mass particles. Hence, even though the boundary surface that encloses these specified mass particles may change considerably over time, the mathematical magnitude of the mass term is a constant. Those mass particles that are included within the mass, or alternately, enclosed by the boundary surface of the mass system, are defined by the analyst as the "mass system under study."

The above basic result, Eq. (1.1), can be derived as follows. Consider a collection of, that is, a specific grouping of, N particles of total mass $m = \sum m_j$, where all such sums run from $j = 1$ to $j = N$, where N can be a very large number. Again, it is not essential that there be any particular geometric relationship between the N particles. Newton's second law applies to each of these N particles. To write Newton's second law in a useful way, let each of these N particles be located by means of its own position vector $\boldsymbol{r}_j(t)$ originating at the origin of a valid coordinate system. Note that if the time-varying spatial position of the ith particle in terms of the valid Cartesian coordinates is $[x_i(t), y_i(t), z_i(t)]$, then the position vector can be written as $\boldsymbol{r}_i(t) = x_i(t)\boldsymbol{i} + y_i(t)\boldsymbol{j} + z_i(t)\boldsymbol{k}$. Since the differential quantity $d\boldsymbol{r}_i$ is tangent to the path of the ith particle, the velocity vector is always tangent to the particle path. However, because the forces applied to the particle are not necessarily tangent to the particle path, neither is the acceleration vector, $d^2\boldsymbol{r}/dt^2$. Thus the path of the particle need not be straight.

The statement of the second law for the individual ith particle now can be written as

$$\boldsymbol{F}_i^{ex} + \boldsymbol{F}_i^{in} = m_i \ddot{\boldsymbol{r}}_i, \tag{1.2}$$

where $\boldsymbol{F}_i^{ex}$ is the vector sum of all the forces acting on the ith particle that originate from sources outside of this collection of N particles (to be called the net external force acting on the ith particle), and $\boldsymbol{F}_i^{in}$ is the vector sum of all the forces acting on ith particle that originate from interactions with the other $N-1$ particles (i.e., the net internal force acting on the ith particle). From Newton's third law, each of the $N-1$ components of the net internal force acting on the ith particle can be associated with an equal and opposite force acting on one of the other particles in the collection of N particles. Hence, summing all such Eqs. (1.2) for the N particles leads to the cancellation of all the internal forces between the N particles, with the result

$$\sum m_j \ddot{\boldsymbol{r}}_j = \sum \boldsymbol{F}_j^{ex} \equiv \boldsymbol{F}^{ex} \equiv \boldsymbol{F}.$$

Again, the total mass m is defined simply as the scalar sum of all the m_i. That is $m = \sum m_j$. The location of the *center of mass* of the total mass m is identified by introducing the center of mass position vector, $\boldsymbol{r}(t)$ (without a subscript). Since this position vector goes from the coordinate origin to the center of mass, this vector alone fully describes the path traveled by the center of mass as a function of time. The center of mass position vector $\boldsymbol{r}$ at any time t is defined so that

$$m\boldsymbol{r} \equiv \sum m_i \boldsymbol{r}_i.$$

This definition means that the center of mass position vector is a mass-weighted average of all the mass particle position vectors. This definition can also be viewed as an application of the mean value theorem. Differentiating both sides of the definition of the center of mass position vector with respect to time twice and then substituting into the previous equation immediately yields Eq. (1.1): $\boldsymbol{F} = m\ddot{\boldsymbol{r}} \equiv m\boldsymbol{a}$. Again, the force vector $\boldsymbol{F}$, without superscripts and subscripts, is the sum of all the external forces. Note that external forces can arise from only one of two sources: (i) the direct contact of the boundary surface of the N particles under study with the boundary of other masses or (ii) the distant action of other masses, in which case they are called field forces. Gravitational forces are an example of the latter type of action.

1.3 Newton's Equations for Rotations

A knowledge of the motion of the center of mass can tell the analyst a lot about the overall motion of the mass system under study. However, that information is incomplete because it tells the analyst nothing at all about the rotations of the mass particles about the center of mass. Since rotational motions can be quite important, this aspect of the overall motion needs investigation.

Just as the translational motion of the center of mass can be viewed as determined by forces, rotational motions are determined by moments of forces. Recall that the mathematical definition of a *moment about a point*, when the moment center is the origin of the valid coordinate system, is

$$\boldsymbol{M} \equiv \boldsymbol{r} \times \boldsymbol{F}.$$

Recall that reversing the order of a vector cross product requires a change in sign to maintain an equality. Further note that it is immaterial where this position vector intercepts the line of action of the above force vector because the product of the magnitude of the $\boldsymbol{r}$ vector and the sine of the angle between the $\boldsymbol{r}$ and $\boldsymbol{F}$ vectors is always equal to the perpendicular distance between (i) the line of action of the force and (ii) the moment center.

Structural engineers are more familiar with moments about Cartesian coordinate axes than moments about points. The relation between a moment about a point and a moment about such an axis can be understood by reference to Figure 1.1(b). This figure illustrates that the moment resulting from the cross product of the $\boldsymbol{r}$ vector and the $\boldsymbol{F}$ vector, by the rules of vector algebra, is in the direction of the unit vector $\boldsymbol{n}$, which is perpendicular to the plane formed by the $\boldsymbol{r}$ and $\boldsymbol{F}$ vectors. The positive direction of $\boldsymbol{n}$ is determined by the thumb of the right hand after sweeping the other four fingers of the right hand from the direction of $\boldsymbol{r}$, the first vector of the cross product, through to the direction of $\boldsymbol{F}$. In terms of α, the angle between these two vectors in the plane formed by the two vectors

$$\boldsymbol{M} \equiv \boldsymbol{r} \times \boldsymbol{F} \equiv Fr \sin\alpha\, \boldsymbol{n}.$$

Like any other vector, the vector $\boldsymbol{M}$ has components along the Cartesian coordinate axes. In terms of the components of the force $\boldsymbol{F}$ and the position vector $\boldsymbol{r}$, the moment about a point can be written, using vector algebra, as follows:

$$\begin{aligned}\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} &= (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \times (F_x\boldsymbol{i} + F_y\boldsymbol{j} + F_z\boldsymbol{k})\\ &= (yF_z - zF_y)\boldsymbol{i} + (zF_x - xF_z)\boldsymbol{j} + (xF_y - yF_x)\boldsymbol{k}\\ &= M_x\boldsymbol{i} + M_y\boldsymbol{j} + M_z\boldsymbol{k}.\end{aligned}$$

Considering the last equation, it is clear that moments about axes are simply components of moments about points.

When describing the rotation of the mass m, it is often convenient to consider a reference point P that is other than the valid coordinate origin, which is here called the point O. See Figure 1.2. Let the this new reference point P move in an arbitrary fashion relative to the coordinate origin, point O, in a fashion defined by the position

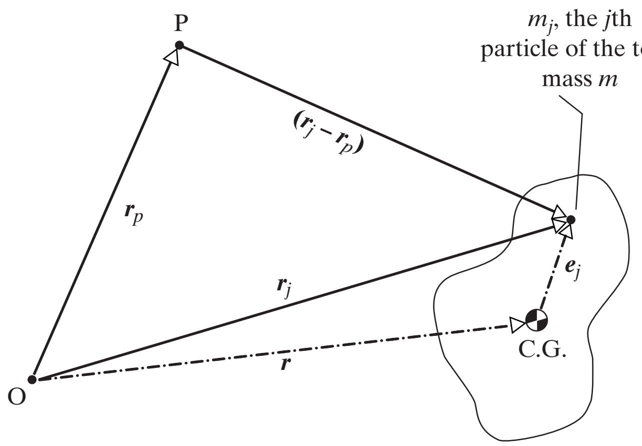

Figure 1.2. Vectors relevant to the rotational motion of a mass. Point P has an arbitrary motion relative to point O.

vector $\boldsymbol{r}_P(t)$. Introduce the vector quantity $\boldsymbol{L}_{Pj}(t)$ which is to be called the *angular momentum* about point P, or, more descriptively, the *moment of momentum* of the mass particle m_j about the arbitrary point P. That is, the angular momentum about point P of the jth mass particle is defined as the vector cross product of (i) the position vector from point P to the particle m_j and (ii) the momentum vector of m_j where the associated velocity vector is that relative to point P rather than the origin of the coordinate system, point O. Thus, in mathematical symbols, relative to point P, the angular momentum of the jth particle, and the angular momentum of the total mass m are, respectively,

$$\boldsymbol{L}_{P_j} \equiv (\boldsymbol{r}_j - \boldsymbol{r}_P) \times m_j(\dot{\boldsymbol{r}}_j - \dot{\boldsymbol{r}}_P) \quad \text{and} \quad \boldsymbol{L}_P \equiv \sum \boldsymbol{L}_{P_j}.$$

Differentiating both sides of the total angular momentum with respect to time, and noting that the cross product of the relative velocity vector $(\dot{\boldsymbol{r}}_j - \dot{\boldsymbol{r}}_P)$ with itself is zero, yields the following result:

$$\frac{d\boldsymbol{L}_P}{dt} = 0 + \sum [(\boldsymbol{r}_j - \boldsymbol{r}_P) \times m_j(\ddot{\boldsymbol{r}}_j - \ddot{\boldsymbol{r}}_P)].$$

From the original statement of Newton's second law, it is possible to substitute in the above equation the net external and internal forces on the jth particle for $m_j(d^2/dt^2)\boldsymbol{r}_j$. The result is

$$\frac{d\boldsymbol{L}_P}{dt} = \sum \left[(\boldsymbol{r}_j - \boldsymbol{r}_P) \times \left(\boldsymbol{F}_j^{ex} + \boldsymbol{F}_j^{in}\right) - m_j(\boldsymbol{r}_j - \boldsymbol{r}_P) \times \ddot{\boldsymbol{r}}_P\right].$$

The term involving the net internal forces sums to zero because all the component internal forces are not only equal and oppositely directed, but, by the strong form of Newton's third law, they are also collinear. See Exercise 1.1. The remaining portion of the first term, that involving the net external forces on the N particles, sums to

$\boldsymbol{M}_P$, called the moment about point P of all the external forces acting on the mass m. The last term in the above sum can be simplified by noting that

$$\begin{aligned}\sum m_i(\boldsymbol{r}_i - \boldsymbol{r}_P) \times \ddot{\boldsymbol{r}}_P &= -\left[\boldsymbol{r}_P \sum m_i - \sum m_i \boldsymbol{r}_i\right] \times \ddot{\boldsymbol{r}}_P \\ &= -[m\boldsymbol{r}_P - m\boldsymbol{r}] \times \ddot{\boldsymbol{r}}_P \\ &= +m(\boldsymbol{r} - \boldsymbol{r}_P) \times \ddot{\boldsymbol{r}}_P.\end{aligned}$$

Thus the final result for the time derivative of the angular momentum of the mass m is

$$\frac{d\boldsymbol{L}_P}{dt} = \boldsymbol{M}_P - m(\boldsymbol{r} - \boldsymbol{r}_P) \times \ddot{\boldsymbol{r}}_P. \tag{1.3a}$$

In other words, with reference to Figure 1.2,

$$\begin{aligned}d\boldsymbol{L}_P/dt = \boldsymbol{M}_P - m &* \text{(position vector from P to the center of mass)} \\ &* \text{(acceleration vector of point P relative to point O)}.\end{aligned}$$

Clearly, if point P is coincident with the center of mass (called the center of mass or CG case, where $\boldsymbol{r}_P = \boldsymbol{r}$), or if the relative position vector $\boldsymbol{r}_P - \boldsymbol{r}$ and the acceleration vector $(d^2/dt2)\boldsymbol{r}_P$ are collinear (unimportant because it is unusual), or if point P is moving at a constant or zero velocity with respect to point O (called, for simplicity, the fixed point or FP case), then the rotation equation reduces to simply

$$\frac{d\boldsymbol{L}_P}{dt} = \boldsymbol{M}_P \qquad \text{if P is a “fixed” point or located at the center of mass.} \tag{1.3b}$$

Note that the above vector equation is the origin of the static equilibrium equations, which state that "the sum of the moments about any *axis* is zero." That is, when the angular momentum relative to the selected *point* P is zero or a constant, then the three orthogonal components of the total moment vector of the external forces acting on the system about point P are zero. These three orthogonal components are the moments about any three orthogonal axes.

The above rotational motion equation, Eq. (1.3b) is not as useful as Eq. (1.1), the corresponding translational motion equation. In Eq. (1.1), the three quantities force, mass, and acceleration are individually quantifiable. In Eq. (1.3b), while the moment term is easily understood, the time rate of change of the angular momentum needs further refinement so that perhaps it too can be written as some sort of fixed mass type of quantity multiplied by some sort of acceleration. Recall that for the mass system m, the total angular momentum relative to point P, is defined as the sum of the moments of the momentum of all the particles that comprise the mass m. That is, again

$$\boldsymbol{L}_P = \sum(\boldsymbol{r}_i - \boldsymbol{r}_P) \times m_i(\dot{\boldsymbol{r}}_i - \dot{\boldsymbol{r}}_P).$$

From the previous development, that is, Eqs. (1.3a,b), there are two simplifying choices for the reference point P: the FP (so-called fixed point) case and the CG (center of mass) case, where the time derivative of the angular momentum is equal to just the moment about point P of all the external forces. First consider the FP case, where point P has only a constant velocity relative to the coordinate origin, point O. Then, from Exercise 1.1, either point P or point O is the origin of a valid Cartesian coordinate system. Since these two points are alike, for the sake of simplicity, let the reference point P coincide with the origin of the coordinate system, point O. Again, this placement of point P at point O does not compromise generality within the FP case because when point P is only moving at a constant velocity relative to point O, point P can also be an origin for a valid coordinate system. Then with $\boldsymbol{r}_P = 0$, and because the $\boldsymbol{e}_i$ vectors of Figure 1.2 originate at the center of mass, the total angular momentum becomes

$$\begin{aligned} \boldsymbol{L}_{FP} &= \sum \boldsymbol{r}_i \times m_i \dot{\boldsymbol{r}}_i = \sum (\boldsymbol{r} + \boldsymbol{e}_i) \times m_i(\dot{\boldsymbol{r}} + \dot{\boldsymbol{e}}_i) \\ &= \boldsymbol{r} \times \dot{\boldsymbol{r}} \left(\sum m_i\right) + \boldsymbol{r} \times \left(\sum m_i \dot{\boldsymbol{e}}_i\right) \\ &\quad + \left(\sum m_i \boldsymbol{e}_i\right) \times \dot{\boldsymbol{r}} + \sum (\boldsymbol{e}_i \times m_i \dot{\boldsymbol{e}}_i) \\ &= \boldsymbol{r} \times m\dot{\boldsymbol{r}} + \sum (\boldsymbol{e}_i \times m_i \dot{\boldsymbol{e}}_i). \end{aligned} \tag{1.4a}$$

To explain why the second and third terms of the above second line are zero, recall the definition of the center of mass position vector, $\boldsymbol{r}$. That mean value definition is $m\boldsymbol{r} \equiv \sum m_i \boldsymbol{r}_i$. Since $\boldsymbol{r}_i = \boldsymbol{r} + \boldsymbol{e}_i$, $m\boldsymbol{r} \equiv \sum m_i \boldsymbol{r} + \sum m_i \boldsymbol{e}_i$. Since $\boldsymbol{r}$ is not affected by the summation over the N particles, it can be factored out of the first sum on the above right-hand side. The result is $m\boldsymbol{r} \equiv m\boldsymbol{r} + \sum m_i \boldsymbol{e}_i$ or $0 = \sum m_i \boldsymbol{e}_i$. Furthermore, because the mass value of each particle is a constant, the time derivative of this last equation shows that $0 = \sum m_i \dot{\boldsymbol{e}}_i$. This is just an illustration of the general fact that first moments, that is, multiplications by distances raised to the first power, of mass or area, or whatever, about the respective mean point are always zero. Multiplications of mass by distances with exponents other than one lead to terms which are generally not zero.

In the above FP equation, Eq. (1.4a), for the angular momentum, the first term depends only on the motion of the center of mass relative to the Cartesian coordinate origin. Even if the mass is not rotating relative to the Cartesian coordinate origin, this term is generally not zero. The second part of the angular momentum exists even if the center of mass is not moving. This second part accounts for the spin of the mass about its own center of mass.

The CG case is where the reference point P is located at the center of mass, point C, rather than at the coordinate origin, point O, as in the FP case. In this CG case, $\boldsymbol{r} = \boldsymbol{r}_P$ and $\boldsymbol{r}_i - \boldsymbol{r}_P = \boldsymbol{e}_i$. Substituting these vector relationships into the expression for $\boldsymbol{L}_P$ immediately leads to the same result for the angular momentum, as was obtained for the FP case, except that the first of those two terms is absent. Hence the mathematics of the CG case are included within that of the FP case, and therefore the CG case does not need a parallel development.

1.4 Simplifications for Rotations

Since Newton's second law is a vector equation, it has been convenient to derive its rotational corollaries by use of vector algebra in three-dimensional space. However, it is no longer convenient to pursue the subject of rotations using three-space vector forms because, in general, the rotations themselves about axes in three dimensions (as opposed to moments about axes in three dimensions) are not vector quantities.

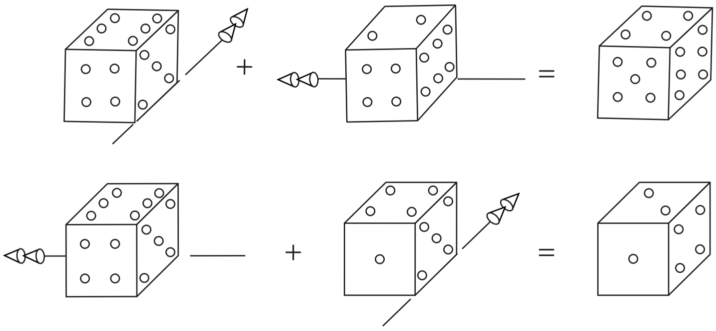

Figure 1.3. Proof that, generally, rotations are not vectors because the order of the rotations is not irrelevant.

For a quantity be classified as a vector, the order of an addition has to be immaterial; that is, it is necessary that $\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{B}+\boldsymbol{A}$, which is called the commutative law for vector addition. In contrast, as Figure 1.3 illustrates, the order of addition of rotations in three-space can greatly change the final orientation of the mass whenever the rotational angles involved are large, like the 90° angles selected for Figure 1.3. There are two simple ways of circumventing this difficulty. The first simplifying approach is to restrict the rotational motion equations to a single plane. In a single plane, all rotations simply add or subtract as scalar quantities. This is a wholly satisfactory approach for most of the illustrative pendulum problems considered in the next chapter. The second option for simplification is to retain rotations about more than one orthogonal axis but limit all those rotations to being small. Here "small" means that the tangent of the angle is closely approximated by the angle itself.[2] As is explained in Ref. [1.2], p. 271, in contrast to larger angles, angles about orthogonal axes of these small magnitudes can be added to each other as vector quantities. This approach of restricting the rotations to either being small or lying in a single plane would not be adequate for formulating a general analysis of the motion of bodies of finite size, which is not a present concern. However, this is a satisfactory approach for almost all structural dynamics problems because structural rotations due solely to the vibrations of a flexible structure are almost always less than 10° or 12°. Therefore, to repeat and thus underline this important point, for the present purposes of structural dynamics, it is often satisfactory only to look at rotations in a single plane or restrict the analysis to small rotations, which can be added vectorially.

To further the discussion, consider all rotations confined to a single plane that, for the sake of explicitness, is identified as the z plane. To reflect the change from three to two dimensions, the notation FP for a fixed point in three-dimensional space, transitions to FA for a fixed axis perpendicular to the z plane. This simplification from a general state of rotations to those only about an axis paralleling the z axis allows the introduction of a pair of convenient unit vectors in the z plane called $\boldsymbol{p}_1$ and $\boldsymbol{q}_1$ such that $\boldsymbol{p}_1$ is directed from the origin toward the center of mass and $\boldsymbol{q}_1$ is rotated 90° counterclockwise from $\boldsymbol{p}_1$. These two unit vectors rotate in the z plane as the center

[2] For example, 10° (expressed in radians) and the tangent of 10° differ by only 1%.

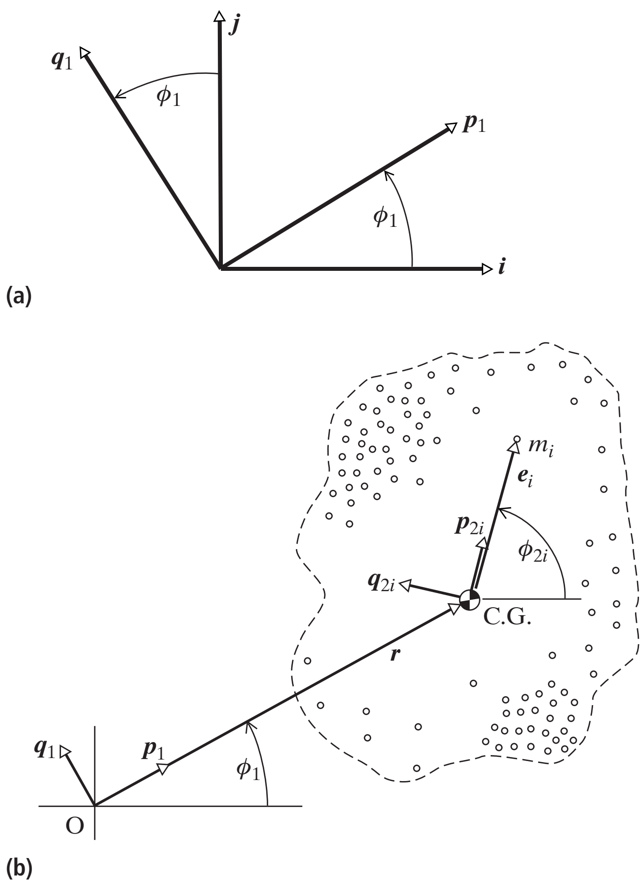

Figure 1.4. (a) The relationship between the rotating unit vectors and the fixed unit vectors, $\boldsymbol{i}$ and $\boldsymbol{j}$. (b) Use of unit vectors to locate the ith mass particle.

of mass moves in that plane. In terms of the fixed-in-space Cartesian coordinate unit vectors, $\boldsymbol{i}$, $\boldsymbol{j}$, as shown in Figure 1.4(a),

$$\boldsymbol{p}_1 = +\boldsymbol{i}\cos\phi_1 + \boldsymbol{j}\sin\phi_1$$
$$\boldsymbol{q}_1 = -\boldsymbol{i}\sin\phi_1 + \boldsymbol{j}\cos\phi_1.$$

Again, even though $\boldsymbol{p}_1$ and $\boldsymbol{q}_1$ have a fixed unit length, they have time derivatives because their orientation in the z plane varies with time as the angle ϕ changes with time. The above equations show that the time derivatives of these rotating unit vectors are

$$\dot{\boldsymbol{p}}_1 = \dot{\phi}_1\boldsymbol{q}_1 \qquad \dot{\boldsymbol{q}}_1 = -\dot{\phi}_1\boldsymbol{p}_1.$$

This unit vector pair $\boldsymbol{p}, \boldsymbol{q}$ can be used with both the position vector for the center of mass and the vector from the center of mass to the ith mass particle. That is, as illustrated in Figure 1.4(b),

$$\boldsymbol{r} = r\,\boldsymbol{p}_1 \quad \text{and} \quad \boldsymbol{e}_i = e_i\,\boldsymbol{p}_{2i}.$$