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Introduction

In this chapter we introduce some concepts and theorems that are important for the rest of this book. Much, but not all, of this material can be found in the book [4]. In general, we have included proofs of theorems only when they do not appear in [4]. The proof of Theorem 1.7.1 is an exception, since we give here a much different proof. We have not included all the basic terminology that we make use of (e.g. graph-theoretic terminology), expecting the reader either to be familiar with such terminology or to consult [4] or other standard references.

1.1 Fundamental Concepts

Let

$$A = [a_{ij}] \quad (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$$

be a matrix of m rows and n columns. We say that A is of *size* m by n, and we also refer to A as an m by n matrix. If $m = n$, then A is a *square* matrix of *order* n. The elements of the matrix A are always real numbers and usually are nonnegative real numbers. In fact, the elements are sometimes restricted to be nonnegative integers, and often they are restricted to be either 0 or 1. The matrix A is composed of m row vectors $\alpha_1, \alpha_2, \ldots, \alpha_m$ and n column vectors $\beta_1, \beta_2, \ldots, \beta_n$, and we write

$$A = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} = [\beta_1 \, \beta_2 \, \ldots \, \beta_n] \, .$$

It is sometimes convenient to refer to either a row or column of the matrix A as a *line* of A. We use the notation A^{T} for the transpose of the matrix A. If $A = A^{\mathrm{T}}$, then A is a square matrix and is *symmetric*.

A zero matrix is always designated by O, a matrix with every entry equal to 1 by J, and an identity matrix by I. In order to emphasize the size of these matrices we sometimes include subscripts. Thus $J_{m,n}$ denotes the m by n matrix of all 1's, and this is shortened to J_n if $m = n$. The notations $O_{m,n}, O_n,$ and I_n have similar meanings.

A *submatrix* of A is specified by choosing a subset of the row index set of A and a subset of the column index set of A. Let $I \subseteq \{1, 2, \ldots, m\}$ and $J \subseteq \{1, 2, \ldots, n\}$. Let $\bar{I} = \{1, 2, \ldots, m\} \setminus I$ denote the *complement* of I in $\{1, 2, \ldots, m\}$, and let $\bar{J} = \{1, 2, \ldots, n\} \setminus J$ denote the complement of J in $\{1, 2, \ldots, n\}$. Then we use the following notations to denote submatrices of A:

$$\begin{aligned}
A[I, J] &= [a_{ij} \colon i \in I, j \in J], \\
A(I, J] &= [a_{ij} \colon i \in \bar{I}, j \in J], \\
A[I, J) &= [a_{ij} \colon i \in I, j \in \bar{J}], \\
A(I, J) &= [a_{ij} \colon i \in \bar{I}, j \in \bar{J}], \\
A[I, \cdot] &= A[I, \{1, 2, \ldots, n\}], \\
A[\cdot, J] &= A[\{1, 2, \ldots, m\}, J], \\
A(I, \cdot] &= A[\bar{I}, \{1, 2, \ldots, n\}], \text{ and} \\
A[\cdot, J) &= A[\{1, 2, \ldots, m\}, \bar{J}].
\end{aligned}$$

These submatrices are allowed to be empty. If $I = \{i\}$ and $J = \{j\}$, then we abbreviate $A(I, J)$ by $A(i, j)$.

We have the following partitioned forms of A:

$$A = \left[\begin{array}{c|c} A[I, J] & A[I, J) \\ \hline A(I, J] & A(I, J) \end{array}\right], \quad A = \left[\begin{array}{c} A[I, \cdot] \\ \hline A(I, \cdot] \end{array}\right],$$

and

$$A = \left[\begin{array}{c|c} A[\cdot, J] & A[\cdot, J) \end{array}\right].$$

The $n!$ permutation matrices of order n are obtained from I_n by arbitrary permutations of its rows (or of its columns). Let $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ be a permutation of $\{1, 2, \ldots, n\}$. Then π corresponds to the permutation matrix $P_\pi = [p_{ij}]$ of order n in which $p_{i\pi_i} = 1$ $(i = 1, 2, \ldots, n)$ and all other $p_{ij} = 0$. The permutation matrix corresponding to the inverse π^{-1} of

π is P_π^{T}. It thus follows that $P_\pi^{-1} = P_\pi^{\mathrm{T}}$, and thus an arbitrary permutation matrix P of order n satisfies the matrix equation

$$PP^{\mathrm{T}} = P^{\mathrm{T}}P = I_n.$$

Let A be a square matrix of order n. Then the matrix PAP^{T} is similar to A. If we let Q be the permutation matrix P^{T}, then $PAP^{\mathrm{T}} = Q^{\mathrm{T}}AQ$. The row vectors of the matrix $P_\pi A$ are $\alpha_{\pi_1}, \alpha_{\pi_2}, \ldots, \alpha_{\pi_m}$. The column vectors of AP_π are $\beta_{\pi'_1}, \beta_{\pi'_2}, \ldots, \beta_{\pi'_n}$ where $\pi^{-1} = (\pi'_1, \pi'_2, \ldots, \pi'_n)$. The column vectors of AP^{T} are $\beta_{\pi_1}, \beta_{\pi_2}, \ldots, \beta_{\pi_n}$. Thus if P is a permutation matrix, the matrix PAP^{T} is obtained from A by *simultaneous permutations of its rows and columns*. More generally, if A is an m by n matrix and P and Q are permutation matrices of orders m and n, respectively, then the matrix PAQ is a matrix obtained from A by *arbitrary permutations of its rows and columns*.

Let $A = [a_{ij}]$ be a matrix of size m by n. The *pattern* (or *nonzero pattern*) of A is the set

$$\mathcal{P}(A) = \{(i,j)\colon a_{ij} \neq 0, \quad i = 1,2,\ldots,m, \quad j = 1,2,\ldots,n\}$$

of positions of A containing a nonzero element.

With the m by n matrix $A = [a_{ij}]$ we associate a combinatorial configuration that depends only on the pattern of A. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a nonempty set of n elements. We call X an *n-set*. Let

$$X_i = \{x_j\colon a_{ij} \neq 0, j = 1,2,\ldots,n\} \quad (i = 1,2,\ldots,m).$$

The collection of m not necessarily distinct subsets $X_1, X_2, \ldots, X_m$ of the n-set X is the *configuration* associated with A. If P and Q are permutation matrices of orders m and n, respectively, then the configuration associated with PAQ is obtained from the configuration associated with A by relabeling the elements of X and reordering the sets $X_1, X_2, \ldots, X_m$. Conversely, given a nonempty configuration $X_1, X_2, \ldots, X_m$ of m subsets of the nonempty n-set $X = \{x_1, x_2, \ldots, x_n\}$, we associate an m by n matrix $A = [a_{ij}]$ of 0's and 1's, where $a_{ij} = 1$ if and only if $x_j \in X_i$ $(i = 1,2,\ldots,m; j = 1,2,\ldots,n)$.

The configuration associated with the m by n matrix $A = [a_{ij}]$ furnishes a particular way to represent the structure of the nonzeros of A. We may view a configuration as a *hypergraph* [1] with vertex set X and *hyperedges* $X_1, X_2, \ldots, X_m$. This hypergraph may have repeated edges, that is, two or more hyperedges may be composed of the same set of vertices. The *edge–vertex incidence matrix* of a hypergraph H with vertex set $X = \{x_1, x_2, \ldots, x_n\}$ and edges $X_1, X_2, \ldots, X_m$ is the m by n matrix $A = [a_{ij}]$ of 0's and 1's in which $a_{ij} = 1$ if and only if x_j is a vertex of edge X_i $(i = 1,2,\ldots,m; j = 1,2,\ldots,n)$. Notice that the hypergraph (configuration) associated with A is the original hypergraph H. If A has exactly two 1's in each row, then A is the edge–vertex incidence matrix of a *multigraph* where a pair of distinct vertices may be joined by more than one edge. If no two rows of A are identical, then this multigraph is a *graph*.

Another way to represent the structure of the nonzeros of a matrix is by a bipartite graph. Let $U = \{u_1, u_2, \ldots, u_m\}$ and $W = \{w_1, w_2, \ldots, w_n\}$ be sets of cardinality m and n, respectively, such that $U \cap W = \emptyset$. The *bipartite graph* associated with A is the graph BG(A) with vertex set $V = U \cup W$ whose edges are all the pairs $\{u_i, w_j\}$ for which $a_{ij} \neq 0$. The pair $\{U, W\}$ is the *bipartition* of BG(A).

Now assume that A is a nonnegative integral matrix, that is, the elements of A are nonnegative integers. We may then associate with A a *bipartite multigraph* BMG(A) with the same vertex set V bipartitioned as above into U and W. In BMG(A) there are a_{ij} edges of the form $\{u_i, w_j\}$ $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$. Notice that if A is a (0,1)-matrix, that is, each entry is either a 0 or a 1, then the bipartite multigraph BMG(A) is a bipartite graph and coincides with the bipartite graph BG(A). Conversely, let BMG be a bipartite multigraph with bipartitioned vertex set $V = \{U, W\}$ where U and W are as above. The *bipartite adjacency matrix* of BMG, abbreviated *bi-adjacency matrix*, is the m by n matrix $A = [a_{ij}]$ where a_{ij} equals the number of edges of the form $\{u_i, w_j\}$ (the *multiplicity* of (u_i, v_j)) $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$. Notice that BMG($A$) is the original bipartite multigraph BMG.

An m by n matrix A is called *decomposable* provided there exist nonnegative integers p and q with $0 < p+q < m+n$ and permutation matrices P and Q such that PAQ is a direct sum $A_1 \oplus A_2$ where A_1 is of size p by q. The conditions on p and q imply that the matrices A_1 and A_2 may be vacuous[1] but each of them contains either a row or a column. The matrix A is *indecomposable* provided it is not decomposable. *The bipartite graph* BG(A) *is connected if and only if A is is indecomposable.*

Assume that the matrix $A = [a_{ij}]$ is square of order n. We may represent its nonzero structure by a digraph $D(A)$. The vertex set of $D(A)$ is taken to be an n-set $V = \{v_1, v_2, \ldots, v_n\}$. There is an arc (v_i, v_j) from v_i to v_j if and only if $a_{ij} \neq 0$ $(i, j = 1, 2, \ldots, n)$. Notice that a nonzero diagonal entry of A determines an arc of $D(A)$ from a vertex to itself (a *directed loop* or *di-loop*). If A is, in addition, a nonnegative integral matrix, then we associate with A a *general digraph* GD(A) with vertex set V where there are a_{ij} arcs of the form (v_i, v_j) $(i, j = 1, 2, \ldots, n)$. If A is a (0,1)-matrix, then GD(A) is a digraph and coincides with $D(A)$. Conversely, let GD be a general digraph with vertex set V. The *adjacency matrix* of GD(A) is the nonnegative integral matrix $A = [a_{ij}]$ of order n where a_{ij} equals the number of arcs of the form (v_i, v_j) (the *multiplicity* of (v_i, v_j)) $(i, j = 1, 2, \ldots, n)$. Notice that GD(A) is the original general digraph GD.

Now assume that the matrix $A = [a_{ij}]$ not only is square but is also symmetric. Then we may represent its nonzero structure by the *graph*

[1]If $p + q = 1$, then A_1 has either a row but no columns or a column but no rows. A similar conclusion holds for A_2 if $p + q = m + n - 1$.

$G(A)$. The vertex set of $G(A)$ is an n-set $V = \{v_1, v_2, \ldots, v_n\}$. There is an edge joining v_i and v_j if and only if $a_{ij} \neq 0$ $(i, j = 1, 2, \ldots, n)$. A nonzero diagonal entry of A determines an edge joining a vertex to itself, that is, a *loop*. The graph $G(A)$ can be obtained from the bipartite graph $\mathrm{BG}(A)$ by identifying the vertices u_i and w_i and calling the resulting vertex v_i $(i = 1, 2, \ldots, n)$. If A is, in addition, a nonnegative integral matrix, then we associate with A a *general graph* $\mathrm{GG}(A)$ with vertex set V where there are a_{ij} edges of the form $\{v_i, v_j\}$ $(i, j = 1, 2, \ldots, n)$. If A is a (0,1)-matrix, then $\mathrm{GG}(A)$ is a graph and coincides with $G(A)$. Conversely, let GG be a general graph with vertex set V. The *adjacency matrix* of GG is the nonnegative integral symmetric matrix $A = [a_{ij}]$ of order n where a_{ij} equals the number of edges of the form (v_i, v_j) (the *multiplicity* of $\{v_i, v_j\}$) $(i, j = 1, 2, \ldots, n)$. Notice that $\mathrm{GG}(A)$ is the original general graph GG. A general graph with no loops is called a *multigraph*.

The symmetric matrix A of order n is *symmetrically decomposable* provided there exists a permutation matrix P such that $PAP^{\mathrm{T}} = A_1 \oplus A_2$ where A_1 and A_2 are both matrices of order at least 1; if A is not symmetrically decomposable, then A is *symmetrically indecomposable*. The matrix A is symmetrically indecomposable if and only if its graph $G(A)$ is connected.

Finally, we remark that if a multigraph MG is bipartite with vertex bipartition $\{U, W\}$ and A is the adjacency matrix of MG, then there are permutation matrices P and Q such that

$$PAQ = \begin{bmatrix} O & C \\ C^{\mathrm{T}} & O \end{bmatrix}$$

where C is the bi-adjacency matrix of MG (with respect to the bipartition $\{U, W\}$).[2]

We shall make use of elementary concepts and results from the theory of graphs and digraphs. We refer to [4], or books on graphs and digraphs, such as [17], [18], [2], [1], for more information.

1.2 Combinatorial Parameters

In this section we introduce several combinatorial parameters associated with matrices and review some of their basic properties. In general, by a *combinatorial property or parameter of a matrix* we mean a property or parameter which is invariant under arbitrary permutations of the rows and columns of the matrix. More information about some of these parameters can be found in [4].

Let $A = [a_{ij}]$ be an m by n matrix. The *term rank* of A is the maximal number $\rho = \rho(A)$ of nonzero elements of A with no two of these elements on a line. The *covering number* of A is the minimal number $\kappa = \kappa(A)$ of

[2] If G is connected, then the bipartition is unique.

lines of A that contain (that is, *cover*) all the nonzero elements of A. Both ρ and κ are combinatorial parameters. The fundamental minimax theorem of König (see [4]) asserts the equality of these two parameters.

Theorem 1.2.1

$$\rho(A) = \kappa(A).$$

A set of nonzero elements of A with no two on a line corresponds in the bipartite graph $\mathrm{BG}(A)$ to a set of edges no two of which have a common vertex, that is, pairwise vertex-disjoint edges or a *matching*. Thus Theorem 1.2.1 asserts that in a bipartite graph, the maximal number of edges in a matching equals the minimal number of vertices in a subset of the vertex set that meets all edges.

Assume that $m \leq n$. The *permanent* of A is defined by

$$\mathrm{per}(A) = \sum a_{1i_1} a_{2i_2} \ldots a_{mi_m}$$

where the summation extends over all sequences $i_1, i_2, \ldots, i_m$ with $1 \leq i_1 < i_2 < \cdots < i_m \leq n$. Thus $\mathrm{per}(A)$ equals the sum of all possible products of m elements of A with the property that the elements in each of the products occur on different lines. The permanent of A is invariant under arbitrary permutations of rows and columns of A, that is,

$$\mathrm{per}(PAQ) = \mathrm{per}(A), \text{if } P \text{ and } Q \text{ are permutation matrices.}$$

If A is a nonnegative matrix, then $\mathrm{per}(A) > 0$ if and only if $\rho(A) = m$. Thus by Theorem 1.2.1, $\mathrm{per}(A) = 0$ if and only if there are permutation matrices P and Q such that

$$PAQ = \begin{bmatrix} A_1 & O_{k,l} \\ A_{21} & A_2 \end{bmatrix}$$

for some positive integers k and l with $k + l = n + 1$. In the case of a square matrix, the permanent function is the same as the determinant function apart from a factor ± 1 preceding each of the products in the defining summation. Unlike the determinant, the permanent is, in general, altered by the addition of a multiple of one row to another and the multiplicative law for the determinant, $\det(AB) = \det(A)\det(B)$, does not hold for the permanent. However, the *Laplace expansion* of the permanent by a row or column does hold:

$$\mathrm{per}(A) = \sum_{j=1}^{n} a_{ij}\mathrm{per}(A(i,j)) \quad (i = 1, 2, \ldots, m);$$

$$\mathrm{per}(A) = \sum_{i=1}^{m} a_{ij}\mathrm{per}(A(i,j)) \quad (j = 1, 2, \ldots, n).$$

We now define the widths and heights of a matrix. In order to simplify the language, we restrict ourselves to $(0,1)$-matrices. Let $A = [a_{ij}]$ be a $(0,1)$-matrix of size m by n with r_i 1's in row i $(i = 1, 2, \ldots, m)$. We call $R = (r_1, r_2, \ldots, r_m)$ the row sum vector of A. Let α be an integer with $0 \leq \alpha \leq r_i$, $i = 1, 2, \ldots, m$. Consider a subset $J \subseteq \{1, 2, \ldots, n\}$ such that each row sum of the m by $|J|$ submatrix

$$E = A[\cdot, J]$$

of A is at least equal to α. Then the columns of E determine an *α-set of representatives* of A. This terminology comes from the fact that in the configuration of subsets $X_1, X_2, \ldots, X_m$ of $X = \{x_1, x_2, \ldots, x_n\}$ associated with A (see Section 1.1), the set $Z = \{x_j : j \in J\}$ satisfies

$$|Z \cap X_i| \geq \alpha \quad (i = 1, 2, \ldots, m).$$

The *α-width* of A equals the minimal number $\epsilon_\alpha = \epsilon_\alpha(A)$ of columns of A that form an α-set of representatives of A. Clearly, $\epsilon_\alpha \geq |\alpha|$, but we also have

$$0 = \epsilon_0 < \epsilon_1 < \cdots < \epsilon_r \tag{1.1}$$

where r is the minimal row sum of A. The widths of A are invariant under row and column permutations.

Let $E = A[\cdot, J]$ be a submatrix of A having at least α 1's in each row and suppose that $|J| = \epsilon(\alpha)$. Then E is a *minimal α-width submatrix* of A. Let F be the submatrix of E composed of all rows of E that contain exactly α 1's. Then F cannot be an empty matrix. Moreover, F cannot have a zero column, because otherwise we could delete the corresponding column of E and obtain an m by $\epsilon_\alpha - 1$ submatrix of A with at least α 1's in each row, contradicting the minimality of ϵ_α. The matrix F is called a *critical α-submatrix* of A. Each critical α-submatrix of A contains the same number ϵ_α of columns, but the number of rows need not be the same. The minimal number $\delta_\alpha = \delta_\alpha(A)$ of rows in a critical α-submatrix of A is called the *α-multiplicity* of A. We observe that $\delta_\alpha \geq 1$ and that multiplicities of A are invariant under row and column permutations. Since a critical α-submatrix cannot contain zero columns, we have $\delta_1 \geq \epsilon(1)$.

Let the matrix A have column sum vector $S = (s_1, s_2, \ldots, s_n)$, and let β be an integer with $0 \leq \beta \leq s_j$ $(1 \leq j \leq n)$. By interchanging rows with columns in the above definition, we may define the *β-height* of A to be the minimal number t of rows of A such that the corresponding t by n submatrix of A has at least β 1's in each column. Since the β-height of A equals the β-width of A^{T}, one may restrict attention to widths.

We conclude this section by introducing a parameter that comes from the theory of hypergraphs [1]. Let A be a (0,1)-matrix of size m by n. A (*weak*) *t-coloring* of A is a partition of its set of column indices into t sets $I_1, I_2, \ldots, I_t$ in such a way that if row i contains more than one 1, then $\{j : a_{ij} = 1\}$ has a nonempty intersection with at least two of the sets $I_1, I_2, \ldots, I_t$ $(i = 1, 2, \ldots, m)$. The sets $I_1, I_2, \ldots, I_t$ are called the *color classes* of the t-coloring. The (*weak*) *chromatic number* $\gamma(A)$ of A is the smallest integer t for which A has a t-coloring [3]. In hypergraph terminology the chromatic number is the smallest number of colors in a coloring of the vertices with the property that no edge with more than one vertex is monochromatic. The *strong chromatic* number of a hypergraph is the smallest number of colors in a coloring of its vertices with the property that no edge contains two vertices of the same color. The *strong chromatic number* $\gamma_s(A)$ of A equals the strong chromatic number of the hypergraph associated with A. If A is the edge–vertex incidence matrix of a graph G, then both the weak and strong chromatic numbers of A equal the chromatic number of G.

1.3 Square Matrices

We first consider a canonical form of a square matrix under simultaneous permutations of its rows and columns.

Let $A = [a_{ij}]$ be a square matrix of order n. Then A is called *reducible* provided there exists a permutation matrix P such that PAP^{T} has the form

$$\begin{bmatrix} A_1 & A_{12} \\ O & A_2 \end{bmatrix}$$

where A_1 and A_2 are square matrices of order at least 1. The matrix A is *irreducible* provided that it is not reducible. A matrix of order 1 is always irreducible. Irreducibility of matrices has an equivalent formulation in terms of digraphs. Let D be a digraph. Then D is *strongly connected* (or *strong*) provided that for each ordered pair of distinct vertices x, y there is a directed path from x to y. A proof of the following theorem can be found in [4].

Theorem 1.3.1 *Let A be a square matrix of order n. Then A is irreducible if and only if the digraph $D(A)$ is strongly connected.*

If a digraph D is not strongly connected, its vertex set can be partitioned uniquely into nonempty sets each of which induces a maximal strong digraph, called a *strong component* of D. This leads to the following canonical form with respect to simultaneous row and column permutations [4].

Theorem 1.3.2 *Let A be a square matrix of order n. Then there exist a permutation matrix P of order n and an integer $t \geq 1$ such that*

$$PAP^{\mathrm{T}} = \begin{bmatrix} A_1 & A_{12} & \ldots & A_{1t} \\ O & A_2 & \ldots & A_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \ldots & A_t \end{bmatrix}. \tag{1.2}$$

where $A_1, A_2, \ldots, A_t$ are square, irreducible matrices. In (1.2)*, the matrices $A_1, A_2, \ldots, A_t$ which occur as diagonal blocks are uniquely determined to within simultaneous permutations of their rows and columns, but their ordering in* (1.2) *is not necessarily unique.* □

Referring to Theorem 1.3.2, we see that the digraph $D(A)$ is composed of strongly connected graphs $D(A_i)$ $(i = 1, 2, \ldots, t)$ and some arcs which go from a vertex in $D(A_i)$ to a vertex in $D(A_j)$ where $i < j$. The matrix A is irreducible if and only if $t = 1$. The matrices $A_1, A_2, \ldots, A_t$ in (1.2) are called the *irreducible components* of A. By Theorem 1.3.2 the irreducible components of A are uniquely determined to within simultaneous permutations of their rows and columns. The matrix A is irreducible if and only if it has exactly one irreducible component.

The canonical form (1.2) of A is given as a block upper triangular matrix, but by reordering the blocks so that the diagonal blocks occur in the order $A_t, \ldots, A_2, A_1$, we could equally well give it as a block lower triangular matrix. Thus *we may interchange the use of lower block triangular and upper block triangular.*

We now consider a canonical form under arbitrary permutations of rows and columns.

Again let A be a square matrix of order n. If $n \geq 2$, then A is called *partly decomposable* provided there exist permutation matrices P and Q such that PAQ has the form

$$\begin{bmatrix} A_1 & A_{12} \\ O & A_2 \end{bmatrix}$$

where A_1 and A_2 are square matrices of order at least 1. According to our definition of decomposable applied to square matrices, the matrix A would be decomposable provided in addition $A_{12} = O$. A square matrix of order 1 is partly decomposable provided it is a zero matrix. The matrix A is *fully indecomposable* provided that it is not partly decomposable. The matrix A is fully indecomposable if and only if it does not have a nonempty zero submatrix $O_{k,l}$ with $k + l \geq n$. Each line of a fully indecomposable matrix of order $n \geq 2$ contains at least two nonzero elements. The covering number $\kappa(A)$ of a fully indecomposable matrix of order n equals n. Moreover, if $n \geq 2$ and we delete a row and column of A, we obtain a matrix of order $n-1$ which has covering number equal to $n-1$. Hence from Theorem 1.2.1 we obtain the following result.

Theorem 1.3.3 *Let A be a fully indecomposable matrix of order n. Then the term rank $\rho(A)$ of A equals n. If $n \geq 2$, then each submatrix of A of order $n-1$ has term rank equal to $n-1$.* □

A collection of n elements (or the positions of those elements) of the square matrix A of order n is called a *diagonal* provided no two of the elements belong to the same row or column; the diagonal is a *nonzero* *diagonal* provided none of its elements equals 0. Thus Theorem 1.3.3 asserts that a fully indecomposable matrix has a nonzero diagonal and, if $n \geq 2$, each nonzero element belongs to a nonzero diagonal.

We have the following canonical form with respect to arbitrary row and column permutations [4].

Theorem 1.3.4 *Let A be a matrix of order n with term rank equal to n. Then there exist permutation matrices P and Q of order n and an integer $t \geq 1$ such that*

$$PAQ = \begin{bmatrix} B_1 & B_{12} & \dots & B_{1t} \\ O & B_2 & \dots & B_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & B_t \end{bmatrix}, \tag{1.3}$$

where $B_1, B_2, \dots, B_t$ are square, fully indecomposable matrices. The matrices $B_1, B_2, \dots, B_t$ which occur as diagonal blocks in (1.3) are uniquely determined to within arbitrary permutations of their rows and columns, but their ordering in (1.3) is not necessarily unique. □

The matrices $B_1, B_2, \dots, B_t$ in (1.3) are called the *fully indecomposable components* of A. By Theorem 1.3.4 the fully indecomposable components of A are uniquely determined to within arbitrary permutations of their rows and columns. The matrix A is fully indecomposable if and only if it has exactly one fully indecomposable component.

As with the canonical form (1.2), the canonical form (1.3) of A is given as a block upper triangular matrix, but by reordering the blocks so that the diagonal blocks occur in the order $B_t, \dots, B_2, B_1$, we could equally well give it as a block lower triangular matrix.

A fully indecomposable matrix of order $n \geq 2$ has an inductive structure which can be formulated in terms of $(0,1)$-matrices (Theorem 4.2.8 in [4]).

Theorem 1.3.5 *Let A be a fully indecomposable $(0,1)$-matrix of order $n \geq 2$. There exist permutation matrices P and Q and an integer $m \geq 2$ such that*

$$PAQ = \begin{bmatrix} A_1 & O & O & \dots & O & E_1 \\ E_2 & A_2 & O & \dots & O & O \\ O & E_3 & A_3 & \dots & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & \dots & A_{m-1} & O \\ O & O & O & \dots & E_m & A_m \end{bmatrix} \tag{1.4}$$

where each of the matrices $A_1, A_2, \dots, A_m$ is fully indecomposable and each of $E_1, E_2, \dots, E_m$ contains at least one 1. □

Note that a matrix of the form (1.4) satisfying the conditions in the statement of the theorem is always fully indecomposable.