1
Introduction

Privately, [Rayleigh] *often quoted with relish a saying attributed to Dalton when in the chair at a scientific meeting: "Well, this is a very interesting paper for those that take any interest in it".*

(Strutt [1157, p. 320])

1.1 What is 'multiple scattering'?

The mathematics of the full treatment may be altogether beyond human power in a reasonable time; nevertheless . . .

(Heaviside [489, p. 324])

'Multiple scattering' means different things to different scientists, but a general definition might be 'the interaction of fields with two or more obstacles'. For example, a typical multiple-scattering problem in classical physics is the scattering of sound waves by two rigid spheres. Further examples, such as the scattering of spherical electron waves by a cluster of atoms, can be found in condensed-matter physics [1379, 168, 422, 424, 423]. Many other examples will be discussed in this book.

The waves scattered by a single obstacle can be calculated in various well-known ways, such as by the method of separation of variables, T-matrix methods or integral-equation methods. All of these methods will be discussed in detail later.

If there are several obstacles, the field scattered from one obstacle will induce further scattered fields from all the other obstacles, which will induce further scattered fields from all the other obstacles, and so on. This recursive way of thinking about how to calculate the total field leads to another notion of multiple scattering; it can be used to actually compute the total scattered field – each step is called an *order of scattering*. In 1893, Heaviside [489, p. 323] gave a clear qualitative description of this 'orders-of-scattering' process.

1.1.1 Single scattering and independent scattering

In his well-known book on electromagnetic scattering, van de Hulst [1233, §1.2] considers two classifications, namely *single scattering* and *independent scattering*. Let us review his definitions of these ideas.

1.1.1.1 Single scattering

This is the simplest approximation, in which the effects of multiple scattering are ignored completely: ‘the total scattered field is just the sum of the fields scattered by the individual [obstacles], each of which is acted on by the [incident] field in isolation from the other [obstacles]’ [111, p. 9]. This approximation is used widely; it is only expected to be valid when the spacing is large compared with both the size of the obstacles and the length of the incident waves. Indeed, with these assumptions, higher-order approximations can be derived [1382, 1383, 1381] and these can be effective [511]. However, there are many instances where multiple scattering is important; for some natural examples, see Bohren’s fascinating book [109] and his related paper [110]. Thus, in atmospheric physics, the single-scattering approximation is not justified, ‘for example, by clouds, where *multiple scattering* can be appreciable’ [111, p. 9].

1.1.1.2 Independent scattering

When waves interact with several obstacles, a ‘cooperative effect’ may occur. This could be constructive interference, leading to unexpectedly large fields, such as can happen with a periodic arrangement of identical scatterers as in a diffraction grating or a crystal lattice. Alternatively, there could be destructive interference, leading to unexpectedly small fields, such as can happen with a random arrangement of scatterers. These are examples of *dependent scattering*: in theory, one ‘has to investigate in detail the phase relations between the waves scattered by neighboring [scatterers]’ [1233, p. 4]. Thus, the ‘assumption of independent scattering implies that there is no systematic relation between these phases’ [1233, p. 5].

The notions of single scattering and independent scattering need not be separated. For example, the authors of [866] consider

> *only independent scattering, randomly positioned particles. This means that particles are separated widely enough, so that each particle scatters light in exactly the same way as if all other particles did not exist. Furthermore, there are no systematic phase relations between partial electromagnetic waves scattered by different particles, so that the intensities . . . of the partial waves can be added without regard to phase. In other words, we will assume that each particle is in the far-field zone of all other particles, and that scattering by different particles is incoherent.*
>
> (Mishchenko et al. [866, p. 4])

The authors go on to quantify what ‘separated widely enough’ means: ‘Exact scattering calculations for randomly oriented two-sphere clusters composed of identical wavelength-sized spheres suggest that particles can scatter independently when the distance between their centers is as small as four times their radius [868]’ [866, p. 5]. This is consistent with van de Hulst [1233, p. 5]: ‘Early estimates have shown that a mutual distance of 3 times the radius is a sufficient condition for independence’.

1.1.2 Scattering by N obstacles

Suppose that we have N disjoint obstacles, B_i, $i = 1, 2, \ldots, N$. The boundary of B_i is S_i. A given wave is incident upon the N obstacles, and the problem is to calculate the scattered waves.

We assume that we know everything about every obstacle: location, shape, orientation and boundary condition; if the obstacles are penetrable, so that waves can travel through them, we assume that we know the internal composition. There are many situations where all of this information is not available; for example, the obstacles might be located randomly.

Mathematically, the exact (deterministic) multiple-scattering problem is easily formulated: it is an exterior boundary-value problem (with a radiation condition at infinity) where the boundary is not simply-connected. However, the problem is not easy to solve, due mainly to the complicated geometry: hence Heaviside's pessimistic comment. Another comment, in a similar vein, was made by van de Hulst [1233]:

> *Multiple scattering does not involve new physical problems, ... Yet the problem of finding the intensities inside and outside the cloud* [of N scatterers] *is an extremely difficult mathematical problem.*
>
> (van de Hulst [1233, p. 6])

This attitude led naturally to single-scattering approximations, as mentioned above. One scatters the incident wave from the ith obstacle (ignoring the presence of the other obstacles), and then sums over i. Indeed, van de Hulst's book and [276] are devoted entirely to single scattering.

At the other extreme, one may attempt to solve the N-body scattering problem directly, perhaps by setting up a boundary integral equation over

$$S = \bigcup_{j=1}^{N} S_j; \tag{1.1}$$

see Chapter 5. Analytically, although 'it would be esthetically preferable to treat the [N-body] configuration as a unit, this approach seems limited to certain special problems' [1196, p. 42]. Computationally, this direct approach can be expensive, especially for problems involving many three-dimensional obstacles.

In the first comprehensive review of the literature on multiple scattering, Twersky opined that

> *it is convenient in considering multiple scattering, to assume that solutions for the component scatterers when isolated are known, and that they may be regarded as "parameters" in the more general problem.*
>
> *Thus, one seeks representations for scattering by many objects in which the effects of the component scatterers are "separated" from the effects of the particular configuration (or statistical distribution of configurations) in the sense that the forms of the results are to hold independently of the type of scatterers involved.*
>
> (Twersky [1198, p. 715])

Similarly, in the context of hydrodynamics (where water waves interact with immersed structures, such as neighbouring ships, wave-power devices or elements of a single larger structure), Ohkusu wrote:

> *For the purpose of calculating hydrodynamic forces . . . , it is essential that only the hydrodynamic properties of each element be given. A method having such a merit will facilitate the calculation for a body having many elements and may be applied to the design arrangement of the elements.*
>
> (Ohkusu [927, p. 107])

In other words, assuming that we know everything about scattering by each obstacle in isolation, how can we use this knowledge to solve the multi-obstacle problem? The best way is to use a 'self-consistent' method. In the next section, we describe such a method in general terms.

1.1.3 Self-consistent methods

A self-consistent method

> *assumes that a wave is emitted by each scatterer of an amount and directionality determined by the radiation incident on that scatterer (the effective field). The latter is to be determined by adding to the incident beam the waves emitted by all other scatterers, and the waves emitted by those scatterers are in turn influenced by the radiation emitted by the scatterer in question. . . . The self-consistent procedure is not an expansion in primary, secondary, tertiary waves, etc. The field acting on a given scatterer, or emitted by it includes the effects of all orders of scattering.*
>
> (Lax [687, pp. 297–298])

Specifically, write the total field as

$$u = u_{\text{inc}} + \sum_{j=1}^{N} u_{\text{sc}}^{j}, \tag{1.2}$$

where u_{inc} is the given incident field and u_{sc}^{j} is the field scattered ('emitted') by the jth scatterer. Define the 'effective' or 'external' or 'exciting field' by

$$u_n \equiv u - u_{\text{sc}}^{n} = u_{\text{inc}} + \sum_{\substack{j=1 \\ j \neq n}}^{N} u_{\text{sc}}^{j}; \tag{1.3}$$

it is the 'radiation incident on [the nth] scatterer' in the presence of all the other scatterers.

Now, as the problem is linear, it must be possible to write

$$u_{\text{sc}}^{j} = \mathcal{T}_j u_j, \tag{1.4}$$

where $\mathcal{T}_j$ is an operator relating the field incident on the jth scatterer, u_j, to the field scattered by the jth scatterer, u_{sc}^j. Hence, (1.3) gives

$$u_n = u_{inc} + \sum_{\substack{j=1 \\ j \neq n}}^{N} \mathcal{T}_j u_j, \qquad n = 1, 2, \ldots, N, \tag{1.5}$$

or, equivalently,

$$u_{sc}^n = \mathcal{T}_n \left\{ u_{inc} + \sum_{\substack{j=1 \\ j \neq n}}^{N} u_{sc}^j \right\}, \qquad n = 1, 2, \ldots, N. \tag{1.6}$$

If one could solve (1.5) for u_n or (1.6) for u_{sc}^n, $n = 1, 2, \ldots, N$, the total field would then be given by

$$u = u_{inc} + \sum_{j=1}^{N} \mathcal{T}_j u_j \tag{1.7}$$

or (1.2), respectively.

The derivation of (1.5) given here follows [1193, Chapter 6, §3]; see also [1191, Chapter 7, §2]. Its simplicity is somewhat illusory, because we have not clearly defined the operator $\mathcal{T}_j$; also, we have not indicated *where* (1.5) or (1.6) is required to hold in space. Nevertheless, we have given an abstract framework within which a variety of concrete methods can be developed.

The general scheme leading to (1.5) and (1.7) is often called the *Foldy–Lax self-consistent method*. Foldy [354] used a special case of the method for ‘isotropic point scatterers’; see Section 8.3 for a detailed description. Lax [687] used the general scheme, with a certain prescription for $\mathcal{T}_j$; see [687, §III]. We will see several specific realisations later, including the T-matrix methods developed in Chapter 7.

For simple geometries, such as circular cylinders or spheres, a self-consistent method is easily realised. One combines separated solutions of the Helmholtz equation (multipoles); a necessary ingredient is an addition theorem for expanding multipoles centred at one origin in terms of similar multipoles centred on a different origin. This old but useful method will be developed in detail in Chapter 4. The method itself goes back to a paper of Lord Rayleigh, published in 1892; we discuss this next.

1.1.4 Rayleigh’s paper of 1892

In his paper ‘On the influence of obstacles arranged in rectangular order upon the properties of a medium’ [1009], Rayleigh considered potential flow through a periodic rectangular array of identical circular cylinders. As a special case of his analysis, let us consider an infinite square array of rigid cylinders of radius a with centres at $(x, y) = (mb, nb)$, where m and n are integers and $b > 2a$; see Fig. 1.1.

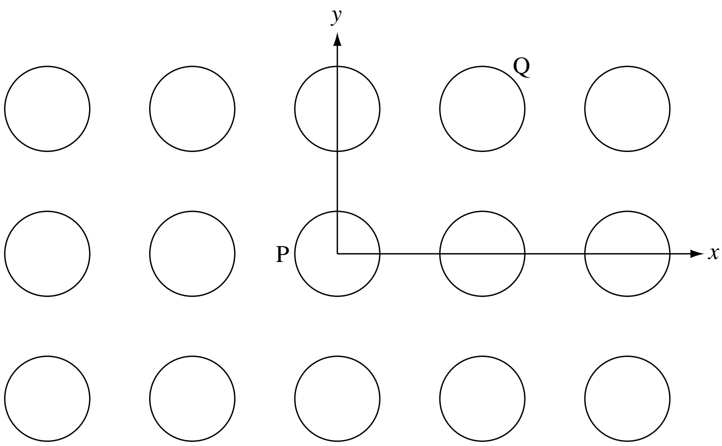

Fig. 1.1. Rayleigh's problem: an infinite square array of circles.

The ambient flow has potential $V = Hx$. In Rayleigh's words [1009, p. 482]: 'If we take the centre of one of the cylinders P as origin of polar coordinates, the potential external to the cylinder may be expanded in the series

$$V = A_0 + (A_1 r + B_1 r^{-1}) \cos\theta + (A_3 r^3 + B_3 r^{-3}) \cos 3\theta + \cdots, \text{'} \tag{1.8}$$

where θ is measured from the x-axis. Symmetry implies that $V - A_0$ must be an odd function of x and an even function of y; these conditions lead to the form of the expansion (1.8). Imposing the boundary condition $\partial V/\partial r = 0$ on $r = a$ gives

$$B_n = a^{2n} A_n, \quad n = 1, 3, 5, \ldots. \tag{1.9}$$

Next [1009, p. 483]: 'The values of the coefficients A_1, B_1, A_3, $B_3 \ldots$ are necessarily the same for all the cylinders, and each may be regarded as a similar multiple source of potential. The first term A_0, however, varies from cylinder to cylinder, as we pass up or down the stream'.

At this stage, we have obtained one condition relating A_n and B_n, namely (1.9), but we need another. To find it, Rayleigh begins as follows [1009, p. 483]: 'The potential V at any point may be regarded as due to external sources at infinity (by which the flow is caused) and to multiple sources situated on the axes of the cylinders. The first part may be denoted by Hx'.

Then, Rayleigh proceeds [1009, p. 484] 'by equating two forms of the expression for the potential at a point x, y near P. The part of the potential due to Hx and to the multiple sources Q (P not included) is

$$A_0 + A_1 r \cos\theta + A_3 r^3 \cos 3\theta + \ldots;$$

or, if we subtract Hx, we may say that the potential at x, y due to multiple sources at Q is the real part of'

$$A_0 + (A_1 - H)z + A_3 z^3 + A_5 z^5 + \cdots, \quad \text{with } z = x + \mathrm{i}y. \tag{1.10}$$

Continuing: 'But if x', y' are the coordinates of the same point when referred to the centre of one of the Q's, the same potential may be expressed by'

$$\Sigma\{B_1 z'^{-1} + B_3 z'^{-3} + \cdots\} \quad \text{with } z' = x' + \mathrm{i}y', \tag{1.11}$$

'the summation being extended over all the Q's. If ξ, η be the coordinates of a Q referred to P, $x' = x - \xi$, $y' = y - \eta$; so that' $B_n z'^{-n} = B_n (z - z_0)^{-n}$ with $z_0 = \xi + \mathrm{i}\eta$. Then, the binomial theorem gives

$$z'^{-n} = (-z_0)^{-n}\left\{1 + n(z/z_0) + \frac{1}{2}n(n+1)(z/z_0)^2 + \cdots\right\}. \tag{1.12}$$

Hence [1009, p. 484]: 'Since (1.10) is the expansion of (1.11) in rising powers of $x + \mathrm{i}y$ $[= z]$, we obtain, equating term to term,'

$$\left.\begin{aligned} H - A_1 &= B_1\Sigma_2 + 3B_3\Sigma_4 + 5B_5\Sigma_6 + \cdots \\ -3!\,A_3 &= 3!\,B_1\Sigma_4 + \frac{1}{2}5!\,B_3\Sigma_6 + \cdots \\ -5!\,A_5 &= 5!\,B_1\Sigma_6 + \frac{1}{2}7!\,B_3\Sigma_8 + \cdots \end{aligned}\right\} \tag{1.13}$$

'and so on, where

$$\Sigma_{2n} = \Sigma(\xi + \mathrm{i}\eta)^{-2n}, ' \tag{1.14}$$

'the summation extending over all the Q's.' (As $\Sigma_{2s+1} = 0$, we also obtain $A_0 = 0$.)

Thus, the system comprising (1.9) and (1.13) can now be solved, in principle, for A_n and B_n. Note that, for two-dimensional potential flow (Laplace's equation), the addition theorem amounts to an application of the binomial theorem, (1.12). Note also that the situation becomes more complicated when the periodicity is destroyed, because then the coefficents A_n and B_n will vary from cylinder to cylinder.

Rayleigh [1009] also considered flow past a rectangular three-dimensional array of identical spheres, and (briefly) low-frequency sound waves through rectangular arrays of rigid cylinders or spheres. For further comments, see [1205, §(6)], [648, §2] and [888, §3.1].

1.1.5 Kasterin, KKR and the electronic structure of solids

Shortly after Rayleigh's paper [1009] *was published, a graduate student at Moscow University, N.P. Kasterin, set out to apply* [Rayleigh's] *ideas to a genuine scattering problem. He chose the relatively simple phenomenon of reflection and refraction of low-frequency sound by an orthorhombic grid of hard* *spheres. ... Kasterin's results were published in his 1903 Moscow thesis. A preliminary report ... came out in 1898.*

(Korringa [648, p. 346])

As a special case of Kasterin's analysis, consider an infinite planar square array of spheres of radius a with centres at $(x, y, z) = (mb, nb, 0)$, where m and n are integers and $b > 2a$. We take the incident field as a plane wave at normal incidence to the array, $u_{\text{inc}} = e^{ikz}$.

Generalising (1.8), the total field at $\boldsymbol{r}$ near the jth sphere can be expanded as

$$u(\boldsymbol{r}) = \sum_{n,m} \left\{ d_{nj}^m \hat{\psi}_n^m(\boldsymbol{r}_j) + c_{nj}^m \psi_n^m(\boldsymbol{r}_j) \right\}, \tag{1.15}$$

where $\boldsymbol{r} = \boldsymbol{r}_j + \boldsymbol{b}_j$ and $\boldsymbol{r} = \boldsymbol{b}_j$ is the sphere's centre. Here, $\psi_n^m(\boldsymbol{r}_j)$ are outgoing multipoles (separated solutions of the Helmholtz equation in spherical polar coordinates), singular at $\boldsymbol{r}_j = \boldsymbol{0}$ $(\boldsymbol{r} = \boldsymbol{b}_j)$ and $\hat{\psi}_n^m(\boldsymbol{r}_j)$ are regular spherical solutions. (Precise definitions will be given later.) The coefficients d_{nj}^m and c_{nj}^m correspond to Rayleigh's A_n and B_n, respectively. Evidently, the periodic geometry and the simple incident field imply that

$$d_{nj}^m \equiv d_n^m \quad \text{and} \quad c_{nj}^m \equiv c_n^m :$$

the coefficients are the same for every sphere.

Applying the boundary condition $\partial u/\partial r = 0$ on $r = a$ yields one relation between d_n^m and c_n^m, namely

$$d_n^m = \gamma_n c_n^m, \tag{1.16}$$

where γ_n is a known constant (see Section 4.6).

The effective field incident on the jth sphere is

$$\sum_{n,m} d_n^m \hat{\psi}_n^m(\boldsymbol{r}_j). \tag{1.17}$$

This must be the same as the sum of the actual incident field and the scattered fields emitted by all the other spheres, namely

$$e^{ikz} + \sum_{\substack{l \\ l \neq j}} \sum_{n,m} c_n^m \psi_n^m(\boldsymbol{r}_l). \tag{1.18}$$

Equating (1.17) and (1.18) in a neighbourhood of the jth sphere gives a second relation between d_n^m and c_n^m. This solves the problem, in principle.

To proceed further, suppose that we have the expansions

$$e^{ikz} = \sum_{n,m} e_n^m \hat{\psi}_n^m(\boldsymbol{r}_j) \tag{1.19}$$

and

$$\psi_n^m(\boldsymbol{r}_l) = \sum_{\nu,\mu} S_{n\nu}^{m\mu}(\boldsymbol{b}_j - \boldsymbol{b}_l)\, \hat{\psi}_\nu^\mu(\boldsymbol{r}_j); \tag{1.20}$$

we will discuss e_n^m and $S_{n\nu}^{m\mu}(\boldsymbol{b})$ shortly. Then, equating (1.17) and (1.18), using (1.16), (1.19) and (1.20), we obtain

$$\gamma_n c_n^m - \sum_{\nu,\mu} c_\nu^\mu \sum_{\substack{l \\ l\neq j}} S_{\nu n}^{\mu m}(\boldsymbol{b}_j - \boldsymbol{b}_l) = e_n^m.$$

In this linear system of algebraic equations, we can take $j = 0$ without loss of generality.

The Rayleigh–Kasterin method, described above, is rigorous, and it can be generalised in various ways. It has been used to obtain numerical solutions for many related problems with (infinite) periodic geometries. For example, see [727] for acoustic scattering by a single periodic row of circles, and see [984] for two-dimensional elastic waves around a square array of circular cavities; see also [888, Chapter 3].

The Rayleigh–Kasterin method was also adapted to problems in solid-state physics. In that context, it is known as the *KKR* (Korringa–Kohn–Rostoker) *method*; see, for example, [1379, §10.3], [775, 424] or [423, §6.8]. For a clear presentation of the two-dimensional KKR method (for sound waves around an infinite square array of soft circles), see [90].

For historical background, including a detailed description of Kasterin's work, see [648].

To generalise Rayleigh's method to a *non-periodic* configuration, consider the problem of acoustic scattering by two spheres (see Section 1.3 for background information). Suppose that the spheres are centred at O_1 and O_2. Write the scattered field u_{sc} as a superposition of outgoing multipoles ψ_n^m, one set singular at O_1 and the other set singular at O_2:

$$u_{\text{sc}} = \sum_{n,m} \{a_n^m \, \psi_n^m(\boldsymbol{r}_1) + b_n^m \, \psi_n^m(\boldsymbol{r}_2)\}.$$

Then, determine the coefficients a_n^m and b_n^m by applying the boundary condition on each sphere in turn: this requires the expansion of $\psi_n^m(\boldsymbol{r}_2)$ in terms of regular spherical solutions centred on O_1, $\hat{\psi}_n^m(\boldsymbol{r}_1)$. Thus, we need the *addition theorem*

$$\psi_n^m(\boldsymbol{r}_2) = \sum_{\nu,\mu} S_{n\nu}^{m\mu}(\boldsymbol{b}) \, \hat{\psi}_\nu^\mu(\boldsymbol{r}_1),$$

which is valid for $r_1 < b$, where $r_1 = |\boldsymbol{r}_1|$, $\boldsymbol{r}_2 = \boldsymbol{r}_1 + \boldsymbol{b}$ and $b = |\boldsymbol{b}|$ is the distance between O_1 and O_2. The matrix $\mathbf{S} = (S_{n\nu}^{m\mu})$ is called the *separation matrix* or the *translation matrix* or the *propagator matrix*. It is an important ingredient in several exact theories of multiple scattering. We will give much attention to various methods for calculating $\mathbf{S}$, with emphasis on acoustic problems (Helmholtz equation) in two (Chapter 2) and three (Chapter 3) dimensions.

We also need expansions of the incident field, similar to (1.19); these will be derived too.

Kasterin did not have explicit expressions for the matrix $\mathbf{S}$: we can see that the expansions (1.19) and (1.20) are analogous to Taylor expansions about $\boldsymbol{r}_j = \mathbf{0}$, and so the coefficients could be obtained by applying appropriate differential operators. This is one of several methods for constructing **S** that we shall develop later.

1.2 Narrowing the scope: previous reviews and omissions

Multiple scattering is a huge subject with a huge literature. For an extensive review up to 1964, see [1198] and the supplement [163]. For a collection of articles surveying many aspects of scattering (including theory, computation and application), see the 957-page volume edited by Pike & Sabatier [977].

There is a 1981 survey by Oguchi on ‘multiple scattering of microwaves or millimeter waves by an assembly of hydrometeors’ [924, p. 719]. Two approaches are reviewed. One is the *Foldy–Lax–Twersky integral equation method*, introduced by Foldy in 1945 [354] and generalised by Lax [687, 688] and Twersky [1200, 1201]. The second approach is based on the *radiative transfer equation*; this may be regarded as the final stage in a larger calculation:

> *the treatment of light scattering by a cloud of randomly positioned, widely separated particles can be partitioned into three steps:* (i) *computation of the far-field scattering and absorption properties of an individual particle . . .* (ii) *computation of the scattering and absorption properties of a small volume element containing a tenuous particle collection by using the single-scattering approximation; and* (iii) *computation of multiple scattering by the entire cloud by solving the radiative transfer equation supplemented by appropriate boundary conditions.*
>
> (Mishchenko *et al.* [870, p. 7])

We do not consider radiative transfer further, but see [1190] for more information.

In 2000, Tourin *et al.* [1182] reviewed a variety of applications, including theory and experiment: in one example of note, sound waves in water are scattered by a random collection of 1000 identical parallel steel rods.

Major areas *not* covered in this book include the following.

(i) Scattering by an infinite number of identical obstacles arranged in some periodic manner, such as in a row or in a regular lattice. For plane-wave scattering, problems of this type can be reduced to a problem in a single ‘unit cell’ (for lattices) or to waveguide problems (for a row of equally-spaced obstacles). The prototype for this reduction, of course, is Rayleigh’s paper [1009], discussed in Section 1.1.4.

Larsen [684] gave an early review of scattering by periodic rows of identical cylinders. For scattering by a semi-infinite periodic row of cylinders, see [859, 502, 501, 861, 729].

(ii) Scattering by an infinite rough surface. For such problems, one has to solve a governing partial differential equation (such as the Helmholtz equation) in the region $y > f(x)$, where $y = f(x)$, $-\infty < x < \infty$, is the rough surface with