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# Hyperbolic Geometry from a Local Viewpoint

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*To Our Families:*

*Jonathan,  
Michael, Erica and Eva,*

*Branko, Milanka,  
Liki and Emily,*

*for their support and  
encouragement.*

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