CAMBRIDGE STUDIES IN
ADVANCED MATHEMATICS 104

NONLINEAR ANALYSIS AND SEMILINEAR ELLIPTIC PROBLEMS

Many problems in science and engineering are described by nonlinear differential equations, which can be notoriously difficult to solve. Through the interplay of topological and variational ideas, methods of nonlinear analysis are able to tackle such fundamental problems. This graduate text explains some of the key techniques in a way that will be appreciated by mathematicians, physicists and engineers. Starting from the elementary tools of bifurcation theory and analysis, the authors cover a number of more modern topics including critical point theory and elliptic partial differential equations. A series of appendices gives convenient accounts of a variety of advanced topics that will introduce the reader to areas of current research. The book is amply illustrated and many chapters are rounded off with a set of exercises.

Cambridge Studies in Advanced Mathematics

All the titles listed below can be obtained from good booksellers or from Cambridge University Press. For a complete series listing visit:
http://www.cambridge.org/mathematics/

Already published

72 F. Borceaux & G. Janelidze *Galois theories*
73 B. Bollobás *Random graphs*
74 R. M. Dudley *Real analysis and probability*
75 T. Sheil-Small *Complex polynomials*
76 C Voisin *Hodge theory and complex algebraic geometry, I*
77 C Voisin *Hodge theory and complex algebraic geometry, II*
78 V. Paulsen *Completely bounded maps and operator algebras*
79 F. Gesztesy & H. Holden *Soliton Equations and Their Algebro-Geometric Solutions, I*
81 S. Mukai *An Introduction to Invariants and Moduli*
82 G. Tourlakis *Lectures in Logic and Set Theory, I*
83 G. Tourlakis *Lectures in Logic and Set Theory, II*
84 R. A. Bailey *Association Schemes*
85 J. Carlson, S. Müller-Stach & C. Peters *Period Mapping and Period Domains*
86 J. J. Duistermaat & J. A. C. Kolk *Multidimensional Real Analysis I*
87 J. J. Duistermaat & J. A. C. Kolk *Multidimensional Real Analysis II*
89 M. Golumbic & A. Trenk *Tolerance Graphs*
90 L. Harper *Global Methods for Combinatorial Isoperimetric Problems*
91 I. Moerdijk & J. Mrcum *Introduction to Foliations and Lie Groupoids*
92 J. Kollar, K. E. Smith & A. Corti *Rational and Nearly Rational Varieties*
93 D. Applebaum *Levy Processes and Stochastic Calculus*
94 B. Conrad *Modular Forms and the Ramanujan Conjecture*
95 M. Schechter *An Introduction to Nonlinear Analysis*
96 R. Carter *Lie Algebras of Finite and Affine Type*
97 H. L. Montgomery, R. C. Vaughan & M. Schechter *Multiplicative Number Theory I*
98 I. Chavel *Riemannian Geometry*
99 D. Goldfeld *Automorphic Forms and L-Functions for the Group GL(n,R)*
100 M. Marcus & J. Rosen *Markov Processes, Gaussian Processes, and Local Times*
101 P. Gille & T. Szamuely *Central Simple Algebras and Galois Cohomology*
102 J. Bertoin *Random Fragmentation and Coagulation Processees*

NONLINEAR ANALYSIS AND SEMILINEAR ELLIPTIC PROBLEMS

ANTONIO AMBROSETTI
ANDREA MALCHIODI

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 2RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521863209

First published 2007

Printed in the United Kingdom at the University Press, Cambridge

A catalogue record for this publication is available from the British Library

ISBN-13 978-0-521-86320-9 hardback
ISBN-10 0-521-86320-1 hardback

Contents

Preface

The main purpose of nonlinear functional analysis is to develop abstract topological and variational methods to study nonlinear phenomena arising in applications. Although this is a rather recent field, initiated about one hundred years ago, remarkable advances have been made and there are now many results that are well established. The fundamental tools of the Leray–Schauder topological degree, local and global bifurcation and critical point theory, can be considered topics that any graduate student in mathematics and physics should know.

This book discusses a selection of the most basic results dealing with the aforementioned topics. The material is presented as simply as possible, in order to highlight the main ideas. In many cases we prefer to state results under slightly stronger assumptions, when this makes the exposition much more clear and avoids some unnecessary technicalities.

The abstract tools are discussed taking into account their applications to semilinear elliptic problems. In some sense, elliptic equations become like a guiding thread, along which the reader will recognize how one method is more suitable than another one, according to the specific feature of the nonlinearity. This is the reason why we discuss both topological methods and variational tools.

After a first chapter containing preliminary material, the book is divided into four parts. The first part is devoted to topological methods and bifurcation theory. Chapter 2 deals with the Lyapunov–Schmidt reduction method and the bifurcation from a simple eigenvalue and connects with the previous book *A Primer of Nonlinear Analysis* [20], of which the present book is a follow up. Chapter 3 deals with the topological degree. First, we define the degree in finite dimension using an analytical approach, which allows us to avoid several technical and cumbersome tools. Next, the Leray–Schauder degree is discussed together with some applications to elliptic boundary value problems.

Among the applications, we also prove the celebrated theorem by Krasnoselski dealing with the bifurcation from an odd eigenvalue for operators of the type *identity-compact*. In Chapter 4 global properties of the degree are discussed. In particular, the global bifurcation result due to Rabinowitz is proved. Special attention is also given to the existence of positive solutions of asymptotically linear boundary value problems.

Parts II and III are devoted to variational methods, namely to critical point theory. After some introductory material presented in Chapters 5 and 6, we discuss in Chapter 7 the main deformation lemmas and the Palais–Smale condition. Chapter 8 deals with the mountain pass and linking theorems. The Lusternik–Schnirelman theory and, in particular, the cases of even functionals on symmetric manifolds are discussed in Chapters 9 and 10, respectively.

Further results on elliptic boundary value problems are presented in Chapter 11, including the pioneering Brezis–Nirenberg result dealing with semilinear equations with critical nonlinearities.

An account of Morse theory is given in Chapter 12 which also contains applications to bifurcation for potential operators and to evaluation of the Morse index of a mountain pass critical point.

Part IV collects a number of appendices which deal with interesting problems that have been left out in the preceding parts because they are more specific in nature, or more complicated, or else because they are objects of current research and therefore are still in evolution. Here our main purpose is to bring the interested reader to the core of contemporary research. In many cases, we are somewhat sketchy, referring to original papers for more details.

Appendix 1 deals with the celebrated Gidas–Ni–Nirenberg symmetry result and with other qualitative results, such as the Liouville type theorem of Gidas and Spruck. Appendix 2 is concerned with the concentration-compactness method introduced by P. L. Lions and includes applications to problems with lack of compactness. Appendix 3 is related to bifurcation theory and deals with bifurcation problems in the absence of compactness, including bifurcation from the essential spectrum. Appendix 4, deals with the classical problem of vortex rings in an ideal fluid. In Appendix 5 we discuss some abstract perturbation methods in critical point theory with their applications to elliptic problems on $\mathbb{R}^n$, to nonlinear Schrödinger equations and to singular perturbation problems. Finally, in Appendix 6 we discuss some problems arising in differential geometry, from the classical Yamabe problem to more recent problems, dealing with fourth order invariants such as the Paneitz curvature.

The book is based on many sources. The first is the material taught in several courses given in past years at SISSA. Some of this material is based on previous lectures delivered by by Giovanni Prodi at the Scuola Normale of Pisa in the 1970s. Very special thanks are due to this great mathematician and friend.

The second source is the papers that we have written on nonlinear analysis. Most of them are works in collaboration with other people: we would like to thank all of them warmly (see the authors of joint papers with A. A. or A. M. listed in the references).

Another input has been discussions with many other friends, including V. Coti Zelati, I. Ekeland, M. Girardi, M. Matzeu and C. Stuart.

A. A. & A. M.