# 1
# Preliminaries

In this chapter we will discuss some preliminary material we will use throughout the book.

## 1.1 Differential calculus

Let us begin with an outline, without proofs, of differential calculus in Banach spaces. For proofs and more details we refer to [20], Chapters 1 and 2.
*The Fréchét derivative.* Let $X, Y$ be Banach spaces and let $L(X, Y)$ denote the space of linear continuous maps from $X$ to $Y$. For $A \in L(X, Y)$ we will often write $Ax$ or $A[x]$ instead of $A(x)$. Endowed with the norm

$$\|A\|_{L(X,Y)} = \sup\{\|Ax\|_Y : \|x\|_X \leq 1\}, \quad A \in L(X, Y),$$

$L(X, Y)$ is a Banach space. If $U \subset X$ is an open set, $C(U, Y)$ denotes the space of continuous maps $f : U \to Y$.

**Definition 1.1** *We say that $f : U \mapsto Y$ is (Fréchét) differentiable at $u \in U$ with derivative* $\mathrm{d}f(u) \in L(X, Y)$ *if*

$$f(u + h) = f(u) + \mathrm{d}f(u)[h] + o(\|h\|), \quad \text{as } h \to 0.$$

*$f$ is said differentiable on $U$ if it is differentiable at every point $u \in U$.*

From the definition it follows that if $f$ is differentiable at $u \in U$ then $f$ is continuous at $u$.

In order to find the derivative of a map $f$ one can evaluate, for all $h \in X$, the limit

$$\lim_{\varepsilon \to 0} \frac{f(u + \varepsilon h) - f(u)}{\varepsilon} \stackrel{\text{def}}{=} A_u h.$$

If $A_u \in L(X, Y)$ and if the map $u \mapsto A_u$ is continuous from $U$ to $L(X, Y)$, then $f$ is differentiable at $u$ and $\mathrm{d}f(u) = A_u$. We will often use $f'(u)$ instead of $\mathrm{d}f(u)$.

Let $f : X \times Y \to Z$, and consider the map $f_v : u \mapsto f(u,v)$, respectively $f_u : v \mapsto f(u,v)$. The *partial derivative* of $f$ with respect to $u$, respectively $v$, at $(u,v) \in X \times Y$ is defined by $\partial_u f(u,v) = \mathrm{d}f_v(u)$, respectively $\partial_v f(u,v) = \mathrm{d}f_u(v)$. In particular, $\partial_u f(u,v) \in L(X,Z)$ and $\partial_v f(u,v) \in L(Y,Z)$. It is easy to see that if $f : X \times Y \mapsto Z$ is differentiable at $(u,v)$, then $f$ is partially differentiable and $\partial_u f(u,v)[h] = \mathrm{d}f_v(u)[h] = \mathrm{d}f(u,v)[h,0]$, respectively $\partial_v f(u,v)[k] = \mathrm{d}f_u(v)[k] = \mathrm{d}f(u,v)[0,k]$. Furthermore, the following result holds.

**Proposition 1.2** *If $f$ possesses the partial derivative with respect to $u$ and $v$ in a neighbourhood $\mathcal{N}$ of $(u,v)$ and the maps $u \mapsto \partial_u f$ and $v \mapsto \partial_v f$ are continuous in $\mathcal{N}$, then $f$ is differentiable at $(u,v)$ and*

$$\mathrm{d}f(u,v)[h,k] = \partial_u f(u,v)[h] + \partial_v f(u,v)[k].$$

In the sequel, if no confusion arises, we will write $f'_u, f'_v$ instead of $\partial_u f$, $\partial_v f$, respectively.

*Higher order derivatives.* Let $f$ be differentiable on $U$ and the map $X \mapsto L(X,Y)$, $u \mapsto \mathrm{d}f(u)$, be differentiable at $u \in U$. The derivative of such a map at $u$ is a second derivative: $\mathrm{d}^2 f(u) \in L(X, L(X,Y))$. From the canonical isomorphism between $L(X,L(X,Y))$ and $L_2(X,Y)$, the space of bilinear maps from $X$ to $Y$, we can and will consider $\mathrm{d}^2 f(u)$ belonging to $L_2(X,Y)$. By induction on $k$ we can define the $k$th derivative $\mathrm{d}^k f(u) \in L_k(X,Y)$, the space of $k$-linear maps from $E$ into $\mathbb{R}$. If $f$ is $k$ times differentiable at every point of $U$, we say that $f$ is $k$ times differentiable on $U$.

We will use the following notation.

- $C^k$ *maps.* If $f$ is $k$ times differentiable on $U$ and the application $U \mapsto L_k(X,Y)$, $u \mapsto \mathrm{d}^k f(u)$, is continuous, we say that $f \in C^k(U,Y)$.
- $C^{0,\alpha}$ *maps.* If $f \in C(U,Y)$ satisfies

$$\sup\left[\frac{\|f(u)-f(v)\|_Y}{\|u-v\|_X^\alpha} : u,v \in U,\ u \neq v\right] < +\infty,$$

for some $\alpha \in (0,1]$, we say that $f \in C^{0,\alpha}(U,Y)$. If $\alpha < 1$, respectively $\alpha = 1$, these maps are nothing but the *Hölder continuous* maps, respectively *Lipschitz* maps.

- $C^{k,\alpha}$ *maps*. If $f \in C^k(U,Y)$ and $\mathrm{d}^k f(u) \in C^{0,\alpha}(U, L_k(X,Y))$, we say that $f \in C^{k,\alpha}(U,Y)$.

Let $f \in C^k(U,Y)$ and suppose that $u, h \in U$ be such that $u + th \in U$ for all $t \in [0,1]$. Since one has

$$\frac{\mathrm{d}^r}{\mathrm{d}t^r} f(u+th) = \mathrm{d}^r f(u+th)[h]^r, \qquad [h]^r = \underbrace{[h,\dots,h]}_{r \text{ times}}, \qquad r = 1,\dots,k,$$

the Taylor formula for $t \mapsto f(u+th)$ yields

$$f(u+h) = f(u) + \mathrm{d}f(u)[h] + \cdots + \frac{1}{k!}\mathrm{d}^k f(u)[h]^k + o(\|h\|^k).$$

*Local inversion and implicit function theorems.* Let $f \in C(U,Y)$, $u^* \in U$ and $v^* = f(u^*) \in Y$. We say that $f$ is *locally invertible* at $u^*$ if there exist neighbourhoods $U^*$ of $u^*$ and $V^*$ of $v^*$ and a map $g \in C(V^*, U^*)$ such that

$$g(f(u)) = u, \quad \forall\, u \in U^*, \qquad f(g(v)) = v, \quad \forall\, v \in V^*.$$

The map $g$ will be denoted by $f^{-1}$.

**Theorem 1.3 (Local inversion theorem)** *Suppose that $f \in C^1(U,Y)$ and that $\mathrm{d}f(u^*)$ is invertible (as a linear map in $L(X,Y)$). Then $f$ is locally invertible at $u^*$, $f^{-1}$ is of class $C^1$ and*

$$\mathrm{d}f^{-1}(v) = (\mathrm{d}f(u))^{-1}, \quad \forall\, v \in V^*, \quad \textit{where } u = f^{-1}(v).$$

*Furthermore, if $f \in C^k(U,Y)$ then $f^{-1}$ is of class $C^k$, as well.*

Let $T, X$ be Banach spaces, $\Lambda \subset T$, $U \subset X$ be open subsets.

**Theorem 1.4 (Implicit function theorem)** *Let $f \in C^k(\Lambda \times U, Y)$, $k \ge 1$, and let $(\lambda^*, u^*) \in \Lambda \times U$ be such that $f(\lambda^*, u^*) = 0$. If $f'_u(\lambda^*, u^*) \in L(X,Y)$ is invertible, then there exist neighbourhoods $\Lambda^*$ of $\lambda^*$, $U^*$ of $u^*$ and a map $g \in C^k(\Lambda^*, X)$ such that*

$$f(\lambda, u) = 0, \qquad (\lambda, u) \in \Lambda^* \times U^* \Leftrightarrow u = g(\lambda).$$

*Moreover one has that*

$$g'(\lambda) = -(f'_u(p))^{-1} \circ f'_\lambda(p), \quad \textit{where } p = (\lambda, g(\lambda)),\ \lambda \in \Lambda^*.$$

## 1.2 Function spaces

We will deal with *bounded domains* $\Omega$ *contained in the Euclidean n-dimensional space* $\mathbb{R}^n$. We will mainly work in the following spaces of functions $u : \Omega \to \mathbb{R}$:

- $L^p(\Omega)$, Lebesgue spaces with norm $\|\cdot\|_{L^p}$;
- $H^{k,p}(\Omega)$, Sobolev spaces with norm $\|\cdot\|_{H^{k,p}}$;
- $C_0^\infty(\Omega)$, the space of functions $u \in C^\infty(\Omega)$ with compact support in $\Omega$;
- $H_0^1(\Omega)$, the closure of $C_0^\infty(\Omega)$ in $H^{1,2}(\Omega)$.

For functions in $H_0^1(\Omega)$ the *Poincaré inequality* holds:

$$\int_\Omega |u|^2 \mathrm{d}x \le c \int_\Omega |\nabla u|^2 \mathrm{d}x,$$

where $c = c(\Omega)$ is a constant (possibly depending on $\Omega$ but independent of $u$).

As a consequence of the Poincaré inequality it follows that

$$\|u\| = \left( \int_\Omega |\nabla u|^2 \, \mathrm{d}x \right)^{1/2}$$

is a norm equivalent to the standard one $\|u\|_{H_0^1}$.

**Theorem 1.5 (Sobolev embedding theorem)** *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$ and let $k \ge 1$, $1 \le p \le \infty$.*

*(i) If $kp < n$, then $H^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 \le q \le np/(n-kp)$; the embedding is compact provided $1 \le q < np/(n-kp)$.*
*(ii) If $kp = n$, then $H^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 \le q < \infty$, and the embedding is compact.*
*(iii) If $kp > n$, then $H^{k,p}(\Omega) \hookrightarrow C^{0,\alpha}(\Omega)$, where*

$$\alpha = \begin{cases} k - n/p & \text{if } k - n/p < 1 \\ 1 & \text{if } k - n/p > 1. \end{cases}$$

*If $k - n/p = 1$, the embedding holds for every $\alpha \in [0, 1)$.*

When we deal with $H_0^1(\Omega)$ the requirement that $\partial\Omega$ is Lipschitz can be eliminated. For future references let us state explicitely what Theorem 1.5 becomes in such a case. We set

$$2^* = \begin{cases} 2n/(n-2) & \text{if } n > 2 \\ +\infty & \text{if } n = 2. \end{cases}$$

**Theorem 1.6** *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. Then*

*(i) if $n > 2$, then $H^1_0(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 \le q \le 2^*$; the embedding is compact provided $1 \le q < 2^*$;*
*(ii) if $n = 2$, then $H^1_0(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 \le q < \infty$;*
*(iii) if $n < 2$, then $H^{1,p}_0(\Omega) \hookrightarrow C^{0,\alpha}(\Omega)$, where $\alpha = 1 - n/2$.*

## 1.3 Nemitski operators

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$. If $u : \Omega \to \mathbb{R}$ is a measurable real valued function, we can consider the map $u \mapsto f(u)$, where $f(u)$ is the real valued function defined on $\Omega$ by setting

$$f(u)(x) = f(x, u(x)).$$

Such a map is called the *Nemitski operator* associated to $f$ and will be denoted with the same symbol $f$. For a discussion of the continuity and differentiability properties of Nemitski operators we refer to [20], Chapter 1, Section 2. Here we want to recall the following result.

**Theorem 1.7** *Let $\alpha, \beta \ge 1$. Suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies*

**$(f.0)$** *$f(x,t)$ is measurable with respect to $x \in \Omega$ for all $t \in \mathbb{R}$ and is continuous with respect $t \in \mathbb{R}$ for a.e. $x \in \Omega$,*

*and that there exists $a_1 \in L^\beta(\Omega)$ and $a_2 > 0$ such that*

$$|f(x,u)| \le a_1(x) + a_2|u|^{\alpha/\beta} \quad \forall (x,u) \in \Omega \times \mathbb{R}, \quad (\alpha, \beta \ge 1). \tag{1.1}$$

*Then the Nemitski operator $f$ is continuous from $L^\alpha(\Omega)$ to $L^\beta(\Omega)$.*

Condition $(f.0)$ is also called the *Caratheodory condition* and a function $f(x,t)$ satisfying $(f.0)$ is usually called a *Caratheodory function*. In most of the concrete applications we will deal with the functional space $H^1_0 = H^1_0(\Omega)$. In such a case, we shall suppose that $f$ satisfies

**$(f.1)$** there exists $a_1 \in L^{2n/(n+2)}(\Omega)$ and $a_2 > 0$ such that

$$|f(x,u)| \le a_1(x) + a_2|u|^p \quad \forall (x,u) \in \Omega \times \mathbb{R}, \tag{1.2}$$

where $p < 2^* - 1$.

In some cases we will weaken $(f.1)$ requiring

**$(f.1')$** $f$ satisfies (1.2) with $p \le 2^* - 1$.

The class of functions $f$ which are locally Hölder continuous and satisfy $(f.1)$ or $(f.1')$, will be denoted by $\mathbb{F}_p$.

Let us point out that, in many cases, we could deal with functions that satisfy $(f.0)$, instead of being locally Hölder continuous; see, for example, Remark 1.9 below. The advantage of working with the class $\mathbb{F}_p$ is that we get classical solutions of elliptic equations we deal with, not merely weak solutions.

Moreover, in the sequel we also take $n > 2$. If $n = 1, 2$ one uses the stronger forms of the Sobolev embedding theorem and the arguments below require minor changes.

If $(f.1')$ holds then, according to Theorem 1.7, one has that

$$f \in C(L^{2^*}, L^{2n/(n+2)}). \tag{1.3}$$

Moreover, setting

$$F(x,u) = \int_0^u f(x,s)\,\mathrm{d}s,$$

it follows that

$$|F(x,u)| \le a_1|u| + a_2|u|^{p+1},$$

with $p+1 \le 2^*$. Since $H_0^1 \hookrightarrow L^{2^*}$ then $F(\cdot, u(\cdot)) \in L^1$ provided $u \in H_0^1$ and it makes sense to consider the map $\Phi : H_0^1 \to \mathbb{R}$ defined by

$$\Phi(u) = \int_\Omega F(x,u)\,\mathrm{d}x. \tag{1.4}$$

One can show, see Theorem 2.9 of [20], that $\Phi$ is of class $C^1$ on $H_0^1$ and

$$\mathrm{d}\Phi(u)[v] = \int_\Omega f(x,u)v\,\mathrm{d}x.$$

Let us point out that, as remarked before, $f(x,u) \in L^{2n/(n+2)}$ while $v \in H_0^1 \subset L^{2^*}$; hence $f(x,u)v \in L^1$ so that the right hand side of the preceding formula makes sense.

Next, suppose that $f \in \mathbb{F}_p$ with $1 < p < (n+2)/(n-2)$, and let $u_n \to u$, weakly in $H_0^1$. Since $p+1 < 2^*$, the embedding of $H_0^1 \hookrightarrow L^{p+1}$ is compact and thus, up to a subsequence, $u_n \to u$ strongly in $L^{p+1}$. This immediately implies that

$$\Phi(u_n) \to \Phi(u),$$

and shows that $\Phi$ is weakly continuous. Similarly, from Theorem 1.7 we infer that $f(u_n) \to f(u)$ in $L^\alpha(\Omega)$, with $\alpha = (p+1)/p$. Using the Hölder inequality we get

$$|\mathrm{d}\Phi(u_n)[v]-\mathrm{d}\Phi(u)[v]| \leq \left[\int_\Omega |f(x,u_n)-f(x,u)|^\alpha \mathrm{d}x\right]^{1/\alpha} \left[\int_\Omega |v|^{p+1}\mathrm{d}x\right]^{1/(p+1)}$$
$$= \|f(u_n)-f(u)\|_{L^\alpha}\|v\|_{L^{p+1}}.$$

Since $p+1 < 2^*$ we deduce that $\|v\|_{L^{p+1}} \leq c\,\|v\|_{H_0^1}$, where $c>0$ is a constant independent of $v$. In conclusion, we infer that

$$\|\mathrm{d}\Phi(u_n)-\mathrm{d}\Phi(u)\| \leq c\,\|f(u_n)-f(u)\|_{L^\alpha},$$

and this shows that $\mathrm{d}\Phi$ is a compact operator. Let us collect the above results in the following.

**Theorem 1.8** *Suppose that $f \in \mathbb{F}_p$ with $1 < p \leq (n+2)/(n-2)$ and let $\Phi$ be defined on $H_0^1(\Omega)$ by (1.4). Then $\Phi \in C^1(H_0^1, \mathbb{R})$.*
*Furthermore, if $f \in \mathbb{F}_p$ with $1 < p < (n+2)/(n-2)$, then $\Phi$ is weakly continuous and $\mathrm{d}\Phi$ is a compact operator.*

**Remark 1.9** The first (respectively second) statement holds true if we suppose that $f$ satisfies $(f.0)$ and $(f.1')$, respectively $(f.0)$ and $(f.1)$. Furthermore, if $f$ is Lipschitz, respectively of class $C^k$, with respect to $u$ then it is easy to see that $\Phi$ is of class $C^{1,1}$, respectively $C^{k+1}$, on $H_0^1$. ■

## 1.4 Elliptic equations

Consider the linear Dirichlet boundary value problem (BVP in short)

$$\begin{cases} -\Delta u(x) = h(x) & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega \end{cases} \tag{1.5}$$

where $h$ is a given function on $\Omega$. If $h \in L^2(\Omega)$, a weak solution of (1.5) is a function $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v\,\mathrm{d}x = \int_\Omega hv\,\mathrm{d}x, \quad \forall v \in C_0^\infty(\Omega).$$

Hereafter $c$ denotes a possibly different constant independent of $u$.

**Theorem 1.10** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain.*

*(i) If $h \in L^p(\Omega)$, $1 < p < +\infty$, then (1.5) has a unique weak solution $u \in H_0^1(\Omega) \cap H^{2,p}(\Omega)$ such that*

$$\|u\|_{H^{2,p}} \leq c\|h\|_{L^p}.$$

(ii) *(Schauder estimates) If* $\Omega$ *is of class* $C^{2,\alpha}$ *and* $h \in C^{0,\alpha}(\overline{\Omega})$ *then* $u \in C^{2,\alpha}(\overline{\Omega})$ *is a classical solution of* (1.5) and

$$\|u\|_{C^{2,\alpha}} \le c\|h\|_{C^{0,\alpha}}.$$

The statement (i) of the preceding theorem allows us to define a linear selfadjoint operator $K : L^2(\Omega) \to H_0^1(\Omega)$, $h \mapsto K(h) = u$, where $u$ denotes the unique solution of (1.5). $K$ is the *Green* operator of $-\Delta$ on $H_0^1(\Omega)$. Since the embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ is compact, it follows that $K$ is compact as a map from $L^2(\Omega)$ in itself. Similarly, we can consider $K$ as an operator in $X = C^{0,\alpha}(\Omega)$. From (ii) of Theorem 1.10 it follows that $K(X) \subset \{u \in C^{2,\alpha}(\Omega) : u_{|\partial\Omega} = 0\}$ and Ascoli's theorem implies that $K$ is still compact.

**Remark 1.11**

(i) We point out that, here and always in the sequel, we can substitute $-\Delta$ with any uniformly elliptic second order operator with smooth coefficients and in divergence form.

(ii) The Schauder estimates stated before hold true when $-\Delta$ is replaced by any second order uniformly elliptic operator such as

$$-\mathcal{L}u = -\sum a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum b_i(x)\frac{\partial u}{\partial x_i} + c(x)u,$$

where $a_{ij}, b_i, c$ are of class $C^1(\overline{\Omega})$, $c \le 0$ in $\overline{\Omega}$ and $\exists\, \kappa > 0$ such that $\sum a_i j(x)\xi_i\xi_j \ge \kappa|\xi|^2$, $\forall\, x \in \overline{\Omega}$, $\xi \in \mathbb{R}^n$. ■

### 1.4.1 Eigenvalues of linear Dirichlet boundary value problems

Consider the linear eigenvalue problem

$$\begin{cases} -\Delta u(x) = \lambda u(x) & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega. \end{cases} \tag{1.6}$$

From the preceding discussion, it follows that (1.6) is equivalent to $u = \lambda K(u)$, $u \in L^2(\Omega)$ or $u \in C^{0,\alpha}(\Omega)$.

It is convenient to recall the main general properties of operators $A \in L(X)$ of the type *identity-compact*, according to the Riesz–Fredholm theory.

**(RF$_1$)** Ker$(A)$ is finite dimensional, Range$(A)$ is closed and has finite codimension;

**(RF$_2$)** Range$(A) = [\text{Ker}(A^*)]^\perp = \{u \in X : \langle\psi, u\rangle = 0,\ \forall\, \psi \in \text{Ker}(A^*)\}$;

**(RF$_3$)** $\exists\, m \ge 1$ such that Ker$(A^k) = $ Ker$(A^{k+1})$, $\forall\, k \ge m$. Moreover, $\forall\, k \ge m$ one has that Range$(A^{k+1}) = $ Range$(A^k)$, $X = \text{Ker}(A^m) \oplus \text{Range}(A^m)$

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and the restriction of $A$ to Range$(A^m)$ is a linear homeomorphism of Range$(A^m)$ onto itself;

**(RF$_4$)** $\mathrm{Ker}(A) = \{0\} \iff \mathrm{Range}(A) = X$.

The preceding results apply to $A_\lambda = I - \lambda K$, where $K$ is the Green operator of $-\Delta$ with zero Dirichlet boundary conditions.

**Definition 1.12** *A real number* $\lambda$ *such that* $\mathrm{Ker}(A_\lambda) \neq \{0\}$ *is an eigenvalue of* (1.6)*. The integer m such that* (RF$_3$) *holds (with* $A_\lambda = A$*) is called the multiplicity of* $\lambda$*. When the multiplicity is equal to* 1 *we say that the eigenvalue is simple.*

The following result holds.

**Theorem 1.13**

*(i) Equation* (1.6) *has a sequence of eigenvalues* $\lambda_k$ *such that*

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots, \qquad \lambda_k \nearrow +\infty.$$

*(Here we use the convention that multiple eigenvalues are repeated according to their multiplicity.) The first eigenvalue* $\lambda_1$ *is simple and the corresponding eigenfunctions do not change sign in* $\Omega$*. Moreover,* $\lambda_1$ *is the only eigenvalue with this property.*
*We will denote by* $\varphi_1$ *the eigenfunction corresponding to* $\lambda_1$*, such that* $\varphi_1(x) > 0$ *and* $\|\varphi_1\|_{L^2} = 1$*. We will also denote by* $\varphi_i$ *the eigenfunctions corresponding to* $\lambda_i$ *such that*

$$\int_\Omega \varphi_h \varphi_k \mathrm{d}x = \begin{cases} 1 & \text{if } h = k \\ 0 & \text{if } h \neq k. \end{cases}$$

*(ii) There holds*

$$\lambda_1 = \min\left\{ \int_\Omega |\nabla u|^2 \mathrm{d}x \,:\, u \in H_0^1(\Omega),\ \int_\Omega u^2 \mathrm{d}x = 1 \right\}.$$

*(iii) Letting* $W_k = \{u \in H_0^1(\Omega) : \int_\Omega \nabla u \cdot \nabla \varphi_h \mathrm{d}x = 0\}$ *for* $h = 1, \dots, k-1$*, one has that*

$$\lambda_k = \min\left\{ \int_\Omega |\nabla u|^2 \mathrm{d}x \,:\, u \in W_k,\ \int_\Omega u^2 \mathrm{d}x = 1 \right\}.$$

Properties (ii) and (iii) are the *variational characterizations* of eigenvalues, see also Section 5.5. Let us also remark that from (ii) above we can deduce a more precise form of the Poincaré inequality:

$$\lambda_1 \int_\Omega u^2 \mathrm{d}x \leq \int_\Omega |\nabla u|^2 \mathrm{d}x. \tag{1.7}$$

Concerning the nonhomogeneous problem

$$\begin{cases} -\Delta u(x) = \lambda u(x) + h(x) & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega, \end{cases} \tag{1.8}$$

we can use the properties ($RF_2$)–($RF_4$) to deduce the following.

**Theorem 1.14**

*(i) If $\lambda \neq \lambda_k$ for all integer $k \geq 1$, then* (1.8) *has a unique solution for any $h \in L^2(\Omega)$.*

*(ii) Let $\lambda_k$ be an eigenvalue of* (1.6) *and let $V_k$ denote the corresponding kernel. Then, given any $h \in L^2(\Omega)$,* (1.8) *has a unique solution if and only if $\int_\Omega hv\,\mathrm{d}x = 0$, for all $v \in V_k$.*

**Remark 1.15** If we work in Hölder spaces, the results stated in Theorems 1.13 and 1.14 hold with $L^2(\Omega)$ substituted by $C^{0,\alpha}(\Omega)$. ■

Let $a \in L^\infty(\Omega)$ be such that $a(x) \geq 0$ and $a(x) > 0$ in a set of positive measure in $\Omega$. We will denote by $\lambda_k[a]$ the eigenvalues of

$$\begin{cases} -\Delta u(x) = \lambda a(x) u(x) & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega. \end{cases}$$

There exist infinitely many eigenvalues $0 < \lambda_1[a] < \lambda_2[a] \leq \lambda_3[a] \leq \dots$ satisfying properties similar to those listed in Theorem 1.13. In addition, the following properties hold.

**(EP-1)** (*Monotonicity property*) If $a \leq b$ then $\lambda_k[a] \geq \lambda_k[b]$, for all $k \geq 1$; moreover, if $a < b$ in a subset $\Omega' \subset \Omega$ with positive measure, then $\lambda_k[a] > \lambda_k[b]$, for all $k \geq 1$.

**(EP-2)** (*Continuity property*) If $a_m \to a$ in $L^{n/2}(\Omega)$, then $\lambda_k[a_m] \to \lambda_k[a]$, for all $k \geq 1$.

### 1.4.2 Regularity

It is a general fact that weak solutions of the Dirichlet BVP

$$\begin{cases} -\Delta u(x) = f(x, u(x)) & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega \end{cases} \tag{1.9}$$

are indeed classical solutions provided $f \in \mathbb{F}_p$ with $1 < p < (n+2)/(n-2)$.