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978-0-521-86103-8 - Central Simple Algebras and Galois Cohomology  
Philippe Gille and Tamás Szamuely  
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## Central Simple Algebras and Galois Cohomology

This book is the first comprehensive, modern introduction to the theory of central simple algebras over arbitrary fields. Starting from the basics, it reaches such advanced results as the Merkurjev–Suslin theorem. This theorem is both the culmination of work initiated by Brauer, Noether, Hasse and Albert in the 1930s and the starting point of motivic cohomology theory, a domain at the forefront of current research in algebraic geometry and K-theory and the setting of recent spectacular work by Voevodsky, Suslin, Rost and others.

Assuming only a solid background in algebra, but no homological algebra, the book covers the basic theory of central simple algebras, methods of Galois descent and Galois cohomology, Severi–Brauer varieties, residue maps and, finally, Milnor K-theory and K-cohomology. A number of noteworthy additional topics are also covered. The last chapter rounds off the theory by presenting the results in positive characteristic, including the theorem of Bloch–Gabber–Kato. The book is suitable as a textbook for graduate students and as a reference for researchers working in algebra, algebraic geometry or K-theory.

PHILIPPE GILLE is Chargé de Recherches, CNRS, Université de Paris-Sud, Orsay.

TAMÁS SZAMUELY is Senior Research Fellow, Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest.

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# Central Simple Algebras and Galois Cohomology

PHILIPPE GILLE

*CNRS, Université de Paris-Sud, Orsay*

TAMÁS SZAMUELY

*Alfréd Rényi Institute of Mathematics,  
Hungarian Academy of Sciences, Budapest*



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## *Preface*

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This book provides a comprehensive and up-to-date introduction to the theory of central simple algebras over arbitrary fields, emphasizing methods of Galois cohomology and (mostly elementary) algebraic geometry. The central result is the Merkurjev–Suslin theorem. As we see it today, this fundamental theorem is at the same time the culmination of the theory of Brauer groups of fields initiated by Brauer, Noether, Hasse and Albert in the 1930s, and a starting point of motivic cohomology theory, a domain which is at the forefront of current research in algebraic geometry and K-theory – suffice it here to mention the recent spectacular results of Voevodsky, Suslin, Rost and others. As a gentle ascent towards the Merkurjev–Suslin theorem, we cover the basic theory of central simple algebras, methods of Galois descent and Galois cohomology, Severi–Brauer varieties, residue maps and finally, Milnor K-theory and K-cohomology. These chapters also contain a number of noteworthy additional topics. The last chapter of the book rounds off the theory by presenting the results in positive characteristic. For an overview of the contents of each chapter we refer to their introductory sections.

**Prerequisites** The book should be accessible to a graduate student or a non-specialist reader with a solid training in algebra including Galois theory and basic commutative algebra, but no homological algebra. Some familiarity with algebraic geometry is also helpful. Most of the text can be read with a basic knowledge corresponding to, say, the first volume of Shafarevich’s text. To help the novice, we summarize in an appendix the results from algebraic geometry we need. The first three sections of Chapter 8 require some familiarity with schemes, and in the proof of one technical statement we are forced to use techniques from Quillen K-theory. However, these may be skipped in a first reading by those willing to accept some ‘black boxes’.

## *Acknowledgments*

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Parts of the book formed the basis of a graduate course by the first author at Université de Paris-Sud and of a lecture series by the two authors at the Alfréd Rényi Institute. We thank both audiences for their pertinent questions and comments, and in particular Endre Szabó who shared his geometric insight with us. Most of the book was written while the first author visited the Rényi Institute in Budapest with a Marie Curie Intra-European Fellowship. The support of the Commission and the hospitality of the Institute are gratefully acknowledged. Last but not least, we are indebted to Diana Gillooly for assuring us a smooth and competent publishing procedure.