

Michael Robert Herman Memorial Volume

This volume is dedicated to the memory of Michel Herman, who was one of the founding editors of *Ergodic Theory & Dynamical Systems*.

His thinking had a huge influence on the breadth and scope of the journal. He bridged several traditions and schools that had emerged in dynamics. His vision and mathematical rigour was essential in making the journal into what it is today. He is sorely missed.

This Memorial Volume was edited by
A. Fathi and J.-C. Yoccoz

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Michel Herman during his lecture for the *International Conference on Dynamical Systems*, IMPA, Rio de Janeiro, 19–28 July 2000, dedicated to the 60th anniversary of Professor Jacob Palis.

Michael Robert Herman, 1942–2000

The untimely death of Michael Robert Herman in November 2000 deprived the scientific community of one of its deepest mathematical minds, who had a profound impact on the theory of dynamical systems over the last 30 years.

Born in New York, he was educated in France. He was a student at École Polytechnique before being one of the first members of the Centre de Mathématiques created there by Laurent Schwartz. For more than 20 years, his seminar had a major influence worldwide and was the main vector of the development of the theory of dynamical systems in France. All of his students remember with thankfulness and emotion the passion with which he led them into the wonderful mathematical world. He maintained through the years strong connections with the Instituto de Matemática Pura e Aplicada in Rio de Janeiro.

His interests covered most aspects of the modern theory of dynamical systems and much beyond that, from economics to arts and philosophy. However, it is fair to say that from the start the so-called small divisors problems, related in particular to the stability of quasiperiodic motions, were closest to his heart. His epoch-making theorem on the linearization of circle diffeomorphisms [1–4]†, his two volumes [5, 6] on invariant curves for twist diffeomorphisms, which are still the standard reference 20 years later, his very many deep contributions on the existence and geometry of invariant tori all bear witness to that interest [7–10].

His new method [11, 12] for bounding from below Lyapunov exponents through subharmonicity has had deep developments.

He was also a master of the subtle counterexample, as his construction [13, 14] for the smooth Hamiltonian closing lemma testifies.

Hopefully this volume reflects the variety of his interests and the frontier of present research, the drawing of which he contributed decisively.

Michael was also one of the founding editors of *Ergodic Theory & Dynamical Systems*. He helped set up high standards for the journal. We have tried to keep these very high standards in this volume.

A. Fathi
J.-C. Yoccoz

† Numbers refer to the list of Herman's published papers that follow this introduction.

LIST OF ARTICLES PUBLISHED BY MICHAEL ROBERT HERMAN

- [1] Michael-Robert Herman. Le groupe linéaire des espaces de Hilbert de dimension infinie est égal à son groupe des commutateurs. *C. R. Acad. Sci. Paris Sér. A–B*, **273** (1971), A15–A17.
- [2] Michael-Robert Herman. Simplicité du groupe des difféomorphismes de classe C^∞ , isotopes à l'identité, du tore de dimension n . *C. R. Acad. Sci. Paris Sér. A–B*, **273** (1971), A232–A234.
- [3] Michael Herman and Francis Sergeraert. Sur un théorème d'Arnold et Kolmogorov. *C. R. Acad. Sci. Paris Sér. A–B*, **273** (1971), A409–A411.
- [4] M. R. Herman. Intersection complète, algébrique, affine, non singulière, en géométrie algébrique réelle. *Thèse de 3e cycle*, Université Paris Sud, Centre d'Orsay, Mars 1972.
- [5] Michael R. Herman. Sur le groupe des difféomorphismes du tore. *Colloque International sur l'Analyse et la Topologie Différentielle (Colloques Internationaux CNRS, Strasbourg, 1972)*. *Ann. Inst. Fourier (Grenoble)*, **23**(2) (1973), 75–86.
- [6] M. R. Herman. Sur le groupe des difféomorphismes \mathbf{R} -analytiques du tore. *Differential Topology and Geometry (Proc. Colloq., Dijon, 1974) (Lecture Notes in Mathematics, 484)*. Springer, Berlin, 1975, pp. 36–42.
- [7] M. R. Herman. Sur l'algèbre de Lie des champs de vecteurs \mathbf{R} -analytiques du tore. *Differential Topology and Geometry (Proc. Colloq., Dijon, 1974) (Lecture Notes in Mathematics, 484)*. Springer, Berlin, 1975, pp. 43–49.
- [8] Michael-Robert Herman. Sur le groupe des difféomorphismes \mathbf{R} -analytiques de \mathbf{R} . *Nederl. Akad. Wetensch. Proc. Ser. A* **78**. *Indag. Math.* **37**(4) (1975), 351–355.
- [9] Michel R. Herman. Sur la conjugaison des difféomorphismes du cercle à des rotations. *Bull. Soc. Math. France, Mém.* **46** (1976), 181–188. *Journées sur la Géométrie de la Dimension Infinie et ses Applications à l'Analyse et à la Topologie (Université Claude-Bernard-Lyon I, Lyon, 1975)*.
- [10] M. R. Herman. Sur les mesures invariantes. *International Conference on Dynamical Systems in Mathematical Physics (Rennes, 1975)*. *Astérisque* **40** (1976), 103–104.
- [11] Michael-Robert Herman. Conjugaison C^∞ des difféomorphismes du cercle dont le nombre de rotations satisfait à une condition arithmétique. *C. R. Acad. Sci. Paris Sér. A–B*, **282**(10), (1976), Ai, A503–A506.
- [12] M. R. Herman. Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations. *Thèse d'Etat*, Université Paris Sud, Centre d'Orsay, Avril 1976.
- [13] Michael-Robert Herman. L^2 regularity of measurable solutions of a finite difference equation of the circle. *Technical Report*, University of Warwick, May 1976. Reprinted in this volume.
- [14] Michael-Robert Herman. Conjugaison C^∞ des difféomorphismes du cercle pour presque tout nombre de rotation. *C. R. Acad. Sci. Paris. Sér. A–B*, **283**(8) (1976), Aii, A579–A582.
- [15] Michael-Robert Herman. Mesure de Lebesgue et nombre de rotation. *Geometry and Topology (Proc. III Latin Amer. School of Math., Inst. Mat. Pura Aplicada CNPq, Rio de Janeiro, 1976) (Lecture Notes in Mathematics, 597)*. Springer, Berlin, 1977, pp. 271–293.
- [16] Michael-Robert Herman. The Godbillon–Vey invariant of foliations by planes of T^3 . *Geometry and Topology (Proc. III Latin Amer. School of Math., Inst. Mat. Pura Aplicada CNPq, Rio de Janeiro, 1976) (Lecture Notes in Mathematics, 597)*. Springer, Berlin, 1977, pp. 294–307.
- [17] Albert Fathi and Michael R. Herman. Existence de difféomorphismes minimaux. *Dynamical Systems, Vol. I—Warsaw. Astérisque* **49** (1977), 37–59.
- [18] Michael-Robert Herman. Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations. *Publ. Math. Inst. Hautes Études Sci.* **49** (1979), 5–233.
- [19] Michael-Robert Herman. Résultats récents sur la conjugaison différentiable. *Proceedings of the International Congress of Mathematicians (Helsinki, 1978)*. Acad. Sci. Fennica, Helsinki, 1980, pp. 811–820.
- [20] A. Fathi, M.-R. Herman and J.-C. Yoccoz. A proof of Pesin's stable manifold theorem. *Geometric Dynamics (Rio de Janeiro, 1981)*. Springer, Berlin, 1983, pp. 177–215.
- [21] M.-R. Herman. Construction d'un difféomorphisme minimal d'entropie topologique non nulle. *Ergod. Th. & Dynam. Sys.* **1**(1) (1981), 65–76.
- [22] M. Herman and J.-C. Yoccoz. Generalizations of some theorems of small divisors to non-Archimedean fields. *Geometric Dynamics (Rio de Janeiro, 1981)*. Springer, Berlin, 1983, pp. 408–447.
- [23] Michael-R. Herman. Une méthode pour minorer les exposants de Lyapounov et quelques exemples montrant le caractère local d'un théorème d'Arnol'd et de Moser sur le tore de dimension 2. *Comment. Math. Helv.* **58**(3) (1983), 453–502.

- [24] Michael-R. Herman. *Sur les courbes invariantes par les difféomorphismes de l'anneau. Vol. 1.* Société Mathématique de France, Paris, 1983 (with an appendix by Albert Fathi and an English summary).
- [25] M.-R. Herman. Sur les difféomorphismes du cercle de nombre de rotation de type constant. *Conference on Harmonic Analysis in Honor of Antoni Zygmund, Vols. I, II (Chicago, IL, 1981)*. Wadsworth, Belmont, CA, 1983, pp. 708–725.
- [26] Michael-R. Herman. Exemples de fractions rationnelles ayant une orbite dense sur la sphère de Riemann. *Bull. Soc. Math. France* **112**(1) (1984), 93–142.
- [27] Michael-R. Herman. Are there critical points on the boundaries of singular domains? *Comm. Math. Phys.* **99**(4) (1985), 593–612.
- [28] Michael-R. Herman. Majoration du nombre de cycles périodiques pour certaines familles de difféomorphismes du cercle. *An. Acad. Brasil. Ciênc.* **57**(3) (1985), 261–263.
- [29] Michael-R. Herman. Simple proofs of local conjugacy theorems for diffeomorphisms of the circle with almost every rotation number. *Bol. Soc. Brasil. Mat.* **16**(1) (1985), 45–83.
- [30] Michael-R. Herman. *Sur les courbes invariantes par les difféomorphismes de l'anneau. Vol. 2.* Number 144. 1986. With a correction to: *On the Curves Invariant Under Diffeomorphisms of the Annulus, Vol. 1* (French) (*Astérisque* **103–104** (1983), MR 85m:58062).
- [31] Michael-R. Herman. Construction of some curious diffeomorphisms of the Riemann sphere. *J. London Math. Soc.* (2) **34**(2) (1986), 375–384.
- [32] Michael-R. Herman. Recent results and some open questions on Siegel's linearization theorem of germs of complex analytic diffeomorphisms of \mathbb{C}^n near a fixed point. *VIIIth International Congress on Mathematical Physics (Marseille, 1986)*. World Scientific Publishing, Singapore, 1987, pp. 138–184.
- [33] Michael-R. Herman. Existence et non existence de tores invariants par des difféomorphismes symplectiques. *Séminaire sur les Équations aux Dérivées Partielles 1987–1988*. École Polytechnique, Palaiseau, 1988, pp. Exp. No. XIV, 24.
- [34] Michael-R. Herman. Inégalités 'a priori' pour des tores lagrangiens invariants par des difféomorphismes symplectiques. *Publ. Math. Inst. Hautes Études Sci.* **70** (1989/90), 47–101.
- [35] M.-R. Herman. On the dynamics of Lagrangian tori invariant by symplectic diffeomorphisms. *Progress in Variational Methods in Hamiltonian Systems and Elliptic Equations (L'Aquila, 1990)*. Longman Scientific and Technical, Harlow, 1992, pp. 92–112.
- [36] Michael-R. Herman. Dynamics connected with indefinite normal torsion. *Twist Mappings and Their Applications*. Springer, New York, 1992, pp. 153–182.
- [37] Michael-R. Herman. Exemples de flots hamiltoniens dont aucune perturbation en topologie C^∞ n'a d'orbites périodiques sur un ouvert de surfaces d'énergies. *C. R. Acad. Sci. Paris Sér. I Math.* **312**(13) (1991), 989–994.
- [38] Michael-R. Herman. Différentiabilité optimale et contre-exemples à la fermeture en topologie C^∞ des orbites récurrentes de flots hamiltoniens. *C. R. Acad. Sci. Paris Sér. I Math.* **313**(1) (1991), 49–51.
- [39] Michel R. Herman. Examples of compact hypersurfaces in \mathbf{R}^{2p} , $2p \geq 6$, with no periodic orbits. *Hamiltonian Systems with Three or More Degrees of Freedom (S'Agaró, 1995)*. Kluwer Academic, Dordrecht, 1999, p. 126.
- [40] Michael Herman. Some open problems in dynamical systems. *Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998)*, number Extra Vol. II, 1998, pp. 797–808 (electronic).

L^2 regularity of measurable solutions of a finite-difference equation of the circle†

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We show that if φ is a lacunary Fourier series and the equation $\psi(x) - \psi(x + \alpha) = \varphi(x)$, $x \bmod 1$ has a measurable solution ψ , then in fact the equation has a solution in L^2 .

(1) We consider the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and the translations (or rotations) $R_\alpha = x \rightarrow x + \alpha$ ($\alpha \in \mathbb{T}$).

For $1 \leq p \leq +\infty$, let $L^p = L^p(\mathbb{T}, dx, \mathbb{C})$ with the norm $\|\cdot\|_p$. The only measure considered is the Haar measure of \mathbb{T} , $dx = m$. All equalities are to be considered m -almost everywhere.

(2) Let $\varphi \in L^1$ and $\alpha \in \mathbb{T}$; we try to solve

$$\psi - \psi \circ R_\alpha = \varphi \tag{*}$$

with ψ measurable and the equality almost everywhere.

If one supposes that ψ is in L^1 , then by identification of Fourier coefficients if

$$\varphi(x) = \sum_{k \in \mathbb{Z}} \hat{\varphi}(k) e^{2\pi i k x},$$

then one has

$$\psi(x) = \sum_{k \in \mathbb{Z}} \frac{\hat{\varphi}(k)}{1 - e^{2\pi i k \alpha}} e^{2\pi i k x},$$

(with the convention that $0/0 = 0$). (Of course one has $0 = \int_{\mathbb{T}} \varphi(x) dx$).

(3) The case when $\alpha = p/q \pmod{1}$, $(p, q) = 1$. Then a necessary and sufficient condition for measurable solutions to (*) is

$$\sum_{i=0}^{q-1} \varphi \circ R_{i\alpha} = 0. \tag{1}$$

If (1) is satisfied then the equation (*) has solutions just as regular as is φ .

† This work of Michel Herman appeared only as a preprint of the Mathematics Institute, University of Warwick, dated May 1976. It was turned into \TeX format by Claire Desescares. Minor editorial work was done by Albert Fathi.

(4) The case when α is irrational. It is easy (by Fourier series) to construct $\varphi \in L^1$ with $\int_{\mathbb{T}} \varphi(x) dx = 0$ and an irrational α such that the equation (*) has no solution in L^1 . By the ergodicity of R_α , measurable solutions of (*) differ by a constant.

If one looks for solutions of (*) which are only measurable then Anosov has shown that one has necessarily

$$\int_{\mathbb{T}} \varphi(x) dx = 0 \quad (\text{for } \varphi \in L^1).$$

Furthermore, Anosov has constructed $\varphi \in C^\omega(\mathbb{T})$ with $\int_{\mathbb{T}} \varphi(x) dx = 0$ and an irrational α such that

$$\sup_{k \neq 0} \left| \frac{\hat{\varphi}(k)}{1 - e^{2\pi i k \alpha}} \right| = +\infty,$$

but nevertheless the equation (*) has a measurable solution ψ (of course not in L^1) (see [1]).

We will show that the examples of Anosov cannot happen when φ is a lacunary Fourier series.

It is then easy to construct a φ with $\int_{\mathbb{T}} \varphi(x) dx = 0$ and an irrational α such that the equation (*) has no measurable solution ψ (since there is no L^2 solution).

For other examples see [6].

(5) Let $\Lambda_+ = n_i$ be a lacunary sequence of positive integers: $n_0 = 1$ and $n_{n+1}/n_i \geq q > 1$ for all i .

Let $\Lambda = \Lambda_+ \cup \{0\} \cup (-\Lambda_+)$ be the symmetric sequence of integers.

One denotes

$$L_\Lambda^p = \{\varphi \in L^p \mid \hat{\varphi}(n) = 0 \text{ if } n \notin \Lambda\}.$$

One says that $\varphi \in L^1$ is a lacunary Fourier series if there exists a lacunary sequence Λ as above such that $\varphi \in L_\Lambda^1$. Then one has, for all $1 \leq p < +\infty$, $\varphi \in L_\Lambda^p$; and all the norms $\|\cdot\|_p$ are equivalent on L_Λ^2 (see [5]).

(6) We propose to prove the following.

THEOREM. *Let $\varphi \in L_\Lambda^2$ and $\alpha \in \mathbb{T}$. If the equation*

$$(*) \psi - \psi \circ R_\alpha = \varphi$$

has a measurable solution ψ , then the equation has a solution in L_Λ^2 and if $\alpha \in \mathbb{T} - \mathbb{Q}/\mathbb{Z}$ then in fact, by the ergodicity of R_α , $\psi \in L_\Lambda^2$.

To prove the theorem one needs the following lemmas.

(7)

LEMMA. *Let $f : \mathbb{T} \rightarrow \mathbb{T}$ be a bijection preserving the Haar measure m .*

Let K be a measurable set of \mathbb{T} . Let $\epsilon > 0$ and the set of integers

$$A = \{n \in \mathbb{Z} \mid m(K \cap f^n(K)) \geq m(K)^2 - \epsilon\}.$$

The set of integers A is relatively dense: there exists a positive integer k , such that $\{j, \dots, j+k\} \cap A \neq \emptyset$, for all $j \in \mathbb{Z}$.

For a proof see [3, p. 31].

(8)

LEMMA†. Let L^2_Λ be given. There exist constants $C > 0$ and b ($0 < b < 1$) such that if $B \subset \mathbb{T}$ is measurable with $m(B) \geq b$, then for all $\varphi \in L^2_\Lambda$ one has

$$C \left(\int_B |\varphi(x)|^2 dx \right)^{1/2} \geq \|\varphi\|_2.$$

Proof. Let $0 < a < 1$ and $\varphi \in L^2_\Lambda$ with $\|\varphi\|_2 = 1$. Let

$$A(\varphi) \equiv A = \{x \in \mathbb{T} \mid |\varphi(x)| \geq a\}.$$

We have $\|\varphi\|_2^2 = 1 = \int_{\mathbb{T}-A} |\varphi(x)|^2 dx + \int_A |\varphi(x)|^2 dx \leq a^2 + \int_A |\varphi(x)|^2 dx$.

One has by the Hölder inequality

$$1 \leq \|\varphi\|_4(m(A))^{1/4} + a.$$

Since the norms $\|\cdot\|_2$ and $\|\cdot\|_4$ are equivalent on L^2_Λ , one has $\|\cdot\|_4 \leq k\|\cdot\|_2$, k being a constant greater than 1.

It follows that

$$m(A) \geq \left(\frac{1-a}{k} \right)^4; \tag{2}$$

choose

$$b = 1 - \frac{1}{2} \left(\frac{1-a}{k} \right)^4.$$

If $B \subset \mathbb{T}$ with $m(B) \geq b$ and if $\varphi \in L^2_\Lambda$ with $\|\varphi\|_2 = 1$, we have

$$m(A(\varphi) \cap B) \geq \frac{1}{2} \left(\frac{1-a}{k} \right)^4$$

by (2), so

$$\int_B |\varphi(x)|^2 dx \geq \frac{1}{2} a^2 \left(\frac{1-a}{k} \right)^4 = \left(\frac{1}{C} \right)^2.$$

The result follows by

$$C \left(\int_B |\varphi(x)|^2 dx \right)^{1/2} \geq \|\varphi\|_2. \quad \square$$

(9)

LEMMA. Let $\varphi \in L^2$. A necessary and sufficient condition for a $\psi \in L^2$ that verifies $\psi - \psi \circ R_\alpha = \varphi$ to exist is that $\sup_{n \in \mathbb{N}} \|\varphi_n\|_2 < +\infty$ with $\varphi_n = \sum_{i=0}^{n-1} \varphi \circ R_{i\alpha}$.

For the proof see [4]. In fact it results from the more general lemma, which uses the fact that the unit ball of a reflexive Banach space is weakly compact, and the Markov–Kakutani fixed point theorem (affine version).

† I thank Y. Meyer who brought to my attention the fact that Carleson has proved a stronger lemma (unfortunately unpublished): For every B with $m(B) > 0$ there exists $C(m(B), q) > 0$ such that one has the conclusion of the lemma. I thank B. Maurey for the proof proposed.

LEMMA. Let L be a reflexive Banach space of norm $\|\cdot\|$ and $u : L \rightarrow L$ a continuous linear operator. Given $x \in L$, a sufficient condition for the existence of a $y \in L$ satisfying $y - u(y) = x$ to exist is that

$$\sup_{n \in \mathbb{N}} \left\| \sum_{i=0}^{n-1} u^i(x) \right\| < +\infty;$$

the condition is necessary if $\sup_{n \in \mathbb{N}} \|u^n\| < +\infty$.

(10) *Proof of the theorem.* Let L_Λ^2 be given and be determined by item (8) (and that depends on Λ).

Let $\epsilon > 0$ with $(1 - \epsilon)^2 - \epsilon \geq b$.

One starts with a measurable solution of

$$\psi - \psi \circ R_\alpha = \varphi, \tag{*}$$

with $\varphi \in L_\Lambda^2$. There exists a compact set $K \subset \mathbb{T}$ of measure $\geq 1 - \epsilon$, such that $\psi|_K$ is continuous. By (*) one has

$$\psi - \psi \circ R_{n\alpha} = \sum_{i=0}^{n-1} \varphi \circ R_{i\alpha} \equiv \varphi_n.$$

It follows that

$$\left(\int_{K \cap R_{n\alpha}(K)} |\varphi_n(x)|^2 dx \right)^{1/2} \leq 2 \sup_{x \in K} |\psi(x)| < +\infty.$$

Let $A = \{n \in \mathbb{Z} \mid m(K \cap R_{n\alpha}(K)) \geq (1 - \epsilon)^2 - \epsilon \geq b\}$. By item (7), the subset A is a relatively dense sequence of integers, and let k be the integer of (7). Let $B = \{-k, -k + 1, \dots, k\}$. Since $\varphi_n \in L_\Lambda^2$ by (8) one has

$$\sup_{n \in A} \|\varphi_n\|_2 = C_1 < +\infty.$$

Let $C_2 = \sup_{n \in B} \|\varphi_n\|_2 < +\infty$. Since every $n \in \mathbb{Z}$ can be written as $n = n_1 + n_2$ with $n_1 \in A$ and $n_2 \in B$ and if n_1 , and n_2 are positive integers, we have

$$\varphi_{n_1+n_2} = \varphi_{n_1} \circ R_{n_2\alpha} + \varphi_{n_2};$$

finally we deduce that

$$\sup_{n \in \mathbb{Z}} \|\varphi_n\|_2 \leq C_1 + C_2$$

and the theorem results from (9).

(11) From the theorem we deduce the following: if $\varphi \in L_\alpha^2$, α is irrational, and ψ is measurable and satisfies $\psi - \psi \circ R_\alpha = \varphi$, then $\psi \in L^p$ for every $1 \leq p < +\infty$ since ψ is a lacunary Fourier series. In general, $\psi \notin L^\infty$ even if φ is of class C^ω as we will show by a classical example.

Construction of an irrational α . Let $\alpha = 1/(a_1 + (1/(a_2 + \dots)))$ be the continued fraction of an irrational α ($a_i \geq 1, a_i \in \mathbb{N}$).

If p_n/q_n are the convergents of α , one has $q_0 = 1, q_1 = a_1$ and $q_n = a_n q_{n-1} + q_{n-2}$, if $n \geq 2$. If $x \in \mathbb{R}$ and $\|x\|$ is the distance of x to the nearest integer, one has

$$\|q_n \alpha\| < \frac{1}{q_{n+1}} \leq \frac{1}{a_{n+1} q_n}.$$

If one chooses the sequence (a_i) so that it increases sufficiently rapidly, one easily constructs an irrational α such that, for every $n \geq 2$, one has

$$\|q_n \alpha\| \leq e^{-q_n}. \tag{+}$$

Let us remark that, for every irrational α , $(q_{2n})_{n \in \mathbb{N}}$ is a lacunary sequence of positive integer (in fact we have $q_{2n+2}/q_{2n} \geq 2$ and also $q_{2n+1}/q_{2n-1} \geq 2$).

Construction of φ . Let $n \geq 1$ be a sequence of complex numbers satisfying

$$\sum_{n=1}^{\infty} |c_{2n}|^2 < +\infty \quad \text{but} \quad \sum_{n=1}^{\infty} |c_{2n}| = +\infty.$$

Let $\varphi(x) = \sum_{n=1}^{\infty} c_{2n} (1 - e^{2\pi i q_{2n} \alpha}) e^{2\pi i q_{2n} x}$.

If α satisfies (+), then $\varphi \in C^\omega(\mathbb{T}, \mathbb{C})$ (and one has $0 = \int_{\mathbb{T}} \varphi(x) dx$).

Let $\psi(x) = \sum_{n=1}^{\infty} c_{2n} e^{2\pi i q_{2n} x}$; one has $\psi \in L^2$ (and ψ is a lacunary Fourier series). Furthermore, one has

$$\psi - \psi \circ R_\alpha = \varphi.$$

But $\psi \notin L^\infty$, for if this was the case then, since ψ is a lacunary Fourier series, we would have $\sum_{n=1}^{\infty} |c_{2n}| < +\infty$, which is contrary to the choice of the sequence (c_{2n}) (see [5]).

(12) We have shown a proposition in [2] that implies the following remark.

Remark. Let $\varphi : \mathbb{T} \rightarrow \mathbb{R}$ be continuous (but not necessarily lacunary) and α irrational. We suppose that there exists $\psi \in L^\infty$ with $\psi - \psi \circ R_\alpha = \varphi$; then ψ is almost everywhere equal to a continuous function.

REFERENCES

- [1] D. V. Anosov. On an additive functional homology equation connected with an ergodic rotation of the circle. *Translation Math. U.S.S.R. Izvestija* **74** (1973), 1257–1271.
- [2] M. R. Herman. Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations. *Thesis*, Orsay 1976.
- [3] K. E. Petersen. Introductory ergodic theory. *Lecture Notes*, University of North Carolina at Chapel Hill.
- [4] K. E. Petersen. On a series of cosecants related to a problem in ergodic theory. *Compositio Math.* **26** (1973), 313–317.
- [5] A. Zygmund. *Trigonometric Series*. Cambridge University Press, 1959.
- [6] A. Y. Gordon. *Funktsional'ny Analiz Evo Prilozheniya* **9**(4) (1975), 71–72 (in Russian).