Introduction

Controversy about the infinite is more or less ubiquitous in philosophy. There are very few areas of philosophy where questions about the infinite do not arise; and there are very few areas of philosophy where questions about the infinite do arise, and where there is no serious dispute about how those questions should be answered. Furthermore, questions about the infinite are foundational in many areas of philosophy: There are many quite fundamental parts of philosophy in which the most basic questions that arise are concerned with the role of the infinite in those parts of philosophy.

Obviously enough, questions about the infinite arise in the most fundamental parts of logic and philosophy of mathematics. One of the most fundamental observations that one can make about the natural numbers – the numbers that are generated when one starts counting from one, adding one unit each time – is that there is no last or greatest natural number: for any number to which one counts, one can go further by adding more units. This observation is one of the fundamental sources of theorising about the infinite, and its consequences extend far beyond the domain of the philosophy of mathematics.

While there are serious questions about the development of concepts of the infinite in logic and mathematics, there are far more immediate problems about the development and application of concepts of the infinite in other realms. Indeed, there are many puzzles and arguments across a wide range of different subject matters that are intended to make difficulties for the suggestion that concepts of the infinite can find application outside the domains of logic and mathematics.

In chapter 1, we begin with a survey of some puzzle cases that are intended to make trouble for the suggestion that concepts of infinity can find application outside the realm of logic and mathematics. After noting a few elementary distinctions among different kinds of concepts of the infinite, we provide a catalogue of seventeen arguments – and puzzle cases – that have been proposed in order to try to establish the conclusion that it is absurd to suppose that certain concepts of the infinite do find legitimate application outside the realm of logic and mathematics. We shall suppose that these arguments and puzzle cases do present a *prima facie* challenge to those who defend the claim that the relevant concepts of the infinite could find legitimate application outside the realm of logic and mathematics, and we shall go on to consider how this challenge might be met.

However, before we undertake this task, we turn – in chapter 2 – to a brief survey of mathematical treatments of the infinite that are needed for serious philosophical discussions of the infinite. Some of the topics of discussion in chapter 2 – for example, standard theories of number and standard theories of analysis – are required in order to provide an adequate discussion of some of the puzzles mentioned in chapter 1; other topics discussed in chapter 2 – such as set theory, Cantor's paradise, and various kinds of nonstandard number systems – are required only for later chapters in the book. Since the presentation is quite compressed, and the material under consideration is not all straightforward, some readers may prefer to skip over this chapter and to come back to those sections that they find are essential for understanding of later parts of the book. While there is nothing in this chapter that is gratuitous, some readers may decide that they can understand the subsequent material well enough without wrestling with all of the mathematical details.

In chapter 3, we turn to a discussion of the arguments and puzzles that were set out in chapter 1. The aim of this discussion is to consider how someone who wants to maintain that concepts of the infinite could find application outside the realm of logic and mathematics could respond to these arguments and puzzles. While the general view that is defended is that these arguments and puzzles cause no serious difficulties for those who maintain that concepts of the infinite could find application outside the realm of logic and mathematics, the details of the discussion vary considerably from one case to the next. Sometimes, it turns out that an allegedly impossible scenario really is impossible for reasons that are relevant to the question of the application of concepts of infinity outside the realms of logic and mathematics: No one should think that concepts of infinity could have *that* kind of application to the extramathematical realm. Other times, it turns out that an allegedly impossible scenario is impossible for reasons that have nothing to do with the question of the application of concepts of infinity outside the realms of logic and mathematics: There are many different ways in which a scenario that involves infinities can be problematic without in any way impugning the possibility that concepts of the infinite can find application outside the realm of logic and mathematics. And, yet other times, an allegedly impossible scenario turns out not to be impossible at all: If concepts of infinity could have certain kinds of application outside the realm of logic and mathematics, then the world would be a strange and different place – but there is a vast difference between *strange* and *impossible*.

Having completed an initial tilt at questions about the application of concepts of infinity outside the realm of logic and mathematics, we then turn our attention to particular subject matters in which questions about the infinite have particular importance. We begin, in chapter 4, with an examination of the role of concepts of infinity in theories of space and time. Our initial point of departure is Zeno's paradoxes of motion, and some sophisticated questions about measures and metrics that are prompted by consideration of Zeno's paradoxes. The remainder of chapter 4 is taken up with considerations about the role of the concept of infinity in more contemporary theorising about the nature of space, time, and spacetime. We consider the arguments from Kant's first antinomy of pure reason concerning the finitude or infinitude of space and time; the range of views that one might take about the mereological structure of space and time, focusing in particular on the question of whether there are spatial and temporal atoms (points and instants, respectively); the suggestion that the obtaining of results from the carrying out of infinitely many distinct operations in a finite amount of time might be finessed in general relativistic spacetimes; and some questions about the classification of singularities in general relativistic spacetimes.

In chapter 5, we turn our attention to questions about infinities that arise in the physical sciences. The first part of the chapter is devoted to a discussion of three case studies: one concerning the existence of hotter than infinite temperatures in physically realised systems; a second concerning the role of renormalisation in quantum field theory; and a third that considers the way in which considerations about infinity have borne on attempts to explain why the sky is dark at night. The remainder of the chapter is then taken up with some more general considerations about the ways in which infinities can enter into physical science, including some fundamental considerations that are relevant to attempts to address the question of whether it is possible for there to be physically instantiated infinities of one kind or another.

The next subject taken up – in chapter 6 – is the role of the concept of infinity in theories of probability and decision. The chapter begins with an elementary exposition of orthodox theories of probability and an examination of the different kinds of additivity principles that one might include in a theory of probability. Attention then turns to decision theory and, in particular, to an examination of the various different ways in which considerations about infinity might be introduced into decision theory. After providing some general reasons for supposing that the construction of infinite decision theory is fraught with difficulties, we conclude the chapter with a discussion of three cases – the two-envelope paradox, the St. Petersburg game, and the puzzle of Heaven and Hell – that can be taken to support the contention that there is no acceptable account of reasoning with unbounded utility functions.

Chapter 7 takes up mereology, the general theory of part/whole relations. Our main interest in this chapter is twofold: On the one hand, we consider principles of composition and arguments that are intended to show that we should be wary of unrestricted claims about the existence of infinite fusions; on the other hand, we consider some claims about infinite divisibility and the existence of mereological atoms. Other topics addressed in this chapter include the arguments in the second Kantian antinomy of pure reason, the question of the bearing of considerations about vagueness on arguments for the existence of infinite collections, some reasons for supposing that indistinguishable quantum mechanical particles are not mereological atoms, an argument for the conclusion that there is no such thing as 'the universe', and some further considerations about the connections between the concept of *continuity* and the concept of a *mereological atom*.

Having completed our tour of individual extramathematical subject matters in which questions about infinity are of particular importance, we turn – in chapter 8 – to a consideration of some more fundamental philosophical questions about our understanding of the infinite. We begin with some general philosophical considerations – concerning, for example, distinctions between actual and merely potential infinities – and then move on to a brief survey of philosophies of pure mathematics and, in particular, of attempts to explain how it is that we are able to understand the classical mathematical concept of infinity. After endorsing Lavine's proposal that we arrive at the classical conception of infinity by a process of *extrapolation* from finite mathematics, we move on to a discussion of philosophies of applied mathematics. The chapter concludes with some remarks about Skolem's paradox and the concept of an infinitesimal quantity.

The last chapter of this part of the book takes up some questions about infinite regresses and the uses and abuses of principles of sufficient reason. Part of the aim of this chapter is to exhibit a circle of concepts – *partial sufficient reason*, *completed infinity*, *infinite regress* – whose application to particular cases is a point of serious contention. More generally, this chapter attacks the suggestion that there are acceptable strong principles of sufficient reason and defends the claim that defensible principles of sufficient reason are so weak that it is implausible to suppose that they possess serious metaphysical bite.

While there are some places where the book is polemical – in particular, this is true of chapters 3 and 9 – it should be emphasised that the primary purpose is to gain a clear view of the ways in which the concept of the infinite is used and of the challenges that confront any attempt to gain an adequate understanding of the employment of this concept. Rather than argue for a particular philosophical account of the infinite, I have taken as a primary goal the exhibition of the costs and benefits of adopting any of the rival accounts of the infinite that have been endorsed by contemporary philosophers. When, in subsequent work, I come to consider the role of the concept of infinity in philosophy of religion, I shall be interested in considering how matters stand on *each* of the accounts of the infinite that finds serious support, and not merely in the question of how things stand on the account of the infinite that I am myself most inclined to accept.

1

Beginnings and Puzzles

There are many different ways of organising discussions of the infinite. We begin by distinguishing among different kinds of problems of the infinite:

(1) There are problems of **large** infinities – collections with at least denumerably many members – and there are problems of **small** infinities (or infinitesimals) – quantities that are nonzero and yet smaller in absolute magnitude than any finite quantity.
(2) There are problems of **denumerable** infinities – collections that are, in some sense, equinumerous with the natural numbers – and there are problems of **nondenumerable** infinities – typically, though not always, collections that are, in some sense, equinumerous with the real numbers. Many of the problems about nondenumerable infinities are related to problems about infinitesimals and connect to questions about the understanding of **continuous** quantities.
(3) There are problems about **theoretical** (or abstract) infinities – infinite collections of numbers, sets, propositions, properties, merely possible worlds, or the like – and there are problems about **physical** (or actual, or instantiated) infinities – infinite collections of physical objects, infinite values of physical quantities, and the like. Of course, there are some entities whose classification is problematic, given this distinction: For instance, should we say that spacetime points are physical objects, or should

we classify them as merely theoretical entities? Indeed, more generally, there are serious questions about whether we should suppose that there *are* numbers, sets, propositions, properties, merely possible worlds, spacetime points, and so on.

Many of the best-known problem cases concern **large, denumerable, physical** infinities. The following are some examples.

1. **Al-Ghazali's Problem**: Suppose that past time is infinite, and that the solar system has persisted unchanged throughout that infinite time. Then both Jupiter and the earth have made infinitely many rotations about the sun – that is, they have made the same number of rotations about the sun. And yet, for every rotation that Jupiter makes, the earth makes thirteen rotations – that is, the earth must have made thirteen times as many rotations as Jupiter. But surely, this is a contradiction. And so, surely, we can conclude that it is incoherent to suppose that past time is infinite – and, moreover, surely we can conclude that there cannot be large, denumerable, physical infinities.[1]
2. **Hilbert's Hotel**: Suppose that there is a hotel – Hilbert's Hotel – with an infinite number of rooms, each of which is occupied by one or more guests. Suppose further that a new guest turns up at reception. Then the proprietors of Hilbert's Hotel can easily accommodate the new guest: They just move the guests in room 1 to room 2; the guests in room 2 to room 3; . . . ; the guests in room N to room $N+1$; . . . ; and house the new guest in room 1. But this is absurd! Given that the hotel was already full, there is no way in which any further guests could be accommodated unless already accommodated guests doubled up. Even worse, if infinitely many new guests arrived, it would be possible to accommodate them in the full hotel; just move the guests in room N to room $2N$, for each N, and then accommodate the newly arrived guests in the odd-numbered rooms. And worse again, if the infinitely many people in the odd-numbered rooms check out, then there are still just as many – because infinitely many – people left in the hotel; but if the infinitely many people

[1] Al-Ghazali's problem is mentioned in Craig (1979a: 98) and Mackie (1982: 93). Brown (1965) attributes a very similar argument to Bonaventure.

in rooms 4, 5, 6, . . . , N, $N + 1$, . . . , check out, then the hotel is all but emptied, even though no more people have left than under the previously described circumstances. The absurdities that surface in this story provide compelling reason to suppose that there cannot be large, denumerable, physical infinities.[2]

3. **Craig's Library**: Suppose that there is a library that contains an infinite collection of books, with a distinct natural number printed on the spine of each book. (We shall consider later the question of how this printing task might be realised.) Then the total number of books in the library is equal to (and not greater than) the number of books that have even numbers on their spines and to the number that have prime numbers on their spines (since the total collection of books, the collection of books that have even numbers on their spines, and the collection of books that have prime numbers on their spines is each a denumerably infinite collection). Moreover, it seems that it is impossible to add an extra book to the collection, since there is no distinct number that could be placed on the spine of the extra book. And what happens if all of the odd-numbered books are taken out on loan? Surely, the missing books would occupy an infinite volume of space on the shelves – and yet, when we push the remaining books together, the shelves will remain full! Once again, the absurdities that arise in this story provide good reason for supposing that there cannot be large, denumerable, physical infinities.[3]
4. **Tristram Shandy**: Suppose that Tristram Shandy writes his autobiography at the rate of one day per year – that is, that it takes him one year of writing to cover one day of his life. If it is possible for Tristram Shandy to have been writing from infinity past, then it is possible for Tristram Shandy to have finished his autobiography by now. But there is no way that Tristram Shandy could have completed his autobiography by now. For suppose that there was a day on which he stopped writing. Since it takes

[2] Hilbert's Hotel is introduced in Gamow (1946: 17); Craig (1979a: 84–6) has initiated considerable recent discussion of this case.

[3] Craig's Library is first discussed in Craig (1979a: 82–6). Subsequent discussions include Smith (1987).

him about 365 days to write about one day, the last day that he could have written about was not the day on which he stopped writing. But in order to complete his autobiography, he must write about the day on which he stops writing. So it seems that it could not have been the case that Tristram Shandy has been writing since infinity past; and so it seems that there cannot be large, denumerable, physical infinities.[4]

5. **Counting from Infinity**: A man counts: "10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, done! I've counted all of the natural numbers backwards; I've been counting from infinity past, and I'm finally done. Thank goodness that's over!" Before we can consider the question of whether we have – or could have – reason to believe that what the man says is true, we need to consider the question of whether we so much as understand what the man is saying. Many people have followed Wittgenstein in supposing that what the man says is not so much as intelligible: We cannot give any content to the suggestion that someone might have "counted backwards from infinity". But if that's right, then we have the strongest possible reason for denying that there can be large, denumerable, physical infinities.[5]

Other well-known problem cases involve both **large, denumerable, physical** infinities and **small, denumerable, physical** infinities. The following are some examples.

6. **Infinite Paralysis**: Suppose that Achilles wants to run straight from A to B but that there are infinitely many gods who, unbeknownst to one another (and to Achilles), each have a reason to prevent him from so doing. The first god forms the following intention: If and when Achilles gets halfway, to paralyse him (instantaneously). The second god forms the following intention: If and when Achilles gets one-quarter of the

[4] The first Tristram Shandy puzzle occurs in Russell (1903: 358–60). It is discussed by Diamond (1964), among others. The very different puzzle presented in the main text is due to Craig (1979a: 98). There has been extensive recent discussion of this case. See, e.g., Sorabji (1983), Conway (1984), Small (1986), Smith (1987), Eells (1988), Craig (1991a), Oderberg (2002a; 2002b), and Oppy (2002c; 2003).

[5] I suspect that this puzzle has an ancient origin. Wittgenstein considered various versions of it. There are more recent discussions in: Dretske (1965), Bennett (1971), Craig (1979a), Moore (1990), and Oderberg (2002a).