

Lasers and Electro-optics

Covering a broad range of topics in modern optical physics and engineering, this textbook is invaluable for undergraduate students studying laser physics, optoelectronics, photonics, applied optics, and optical engineering.

This new edition has been re-organized, so that it now covers many new topics such as the optics of stratified media, quantum-well lasers and modulators, free-electron lasers, diode-pumped solid-state and gas lasers, imaging and non-imaging optical systems, squeezed light, periodic poling in nonlinear media, very-short-pulse lasers, and new applications of lasers.

The textbook gives a detailed introduction to the basic physics and engineering of lasers, as well as covering the design and operational principles of a wide range of optical systems and electro-optic devices. It features full details of important derivations and results, and provides many practical examples of the design, construction, and performance characteristics of different types of lasers and electro-optic devices.

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Lasers and Electro-optics

Second Edition

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Preface to the Second Edition

The author of a text generally feels obligated to explain the reasons for his or her writing. This is a matter of tradition as it provides an opportunity for explaining the development and philosophy of the text, its subject matter and intended audience, and acknowledges the help that the author has received. In the case of a second edition of a text, as is the case here, a new preface provides an opportunity for the author to explain the revisions of the second edition and to further acknowledge help from colleagues. I hope to accomplish these tasks briefly here.

The first edition of this text grew over many years out of notes that I had developed for courses at the senior undergraduate and beginning graduate student level at the University of Manchester, Cornell University, and the University of Maryland, College Park. These courses covered many aspects of laser physics and engineering, the practical aspects of optics that pertain to an understanding of these subjects, and a discussion of related phenomena and devices whose importance has grown from the invention of the laser in 1960. These include nonlinear optics, electro-optics, acousto-optics, and the devices that take practical advantage of these phenomena. The names given to the fields that encompass such subject matter have included laser physics, optical electronics, optoelectronics, photonics, and quantum electronics. The fundamentals of these subjects have not changed significantly in the years that have intervened since the publication of the first edition. However, there have been important technological advances that need discussion, as well as new laser applications. A few important new topic areas now included are fiber lasers, aberrations in optical systems, stratified media, photonic crystals, periodic poling for phase matching, and new methods for obtaining laser pulses of very short duration. Expanded discussion is provided about diode-pumped solid-state lasers, free-electron lasers, semiconductor physics, semiconductor lasers, photon statistics, squeezed light, and analysis of polarized light.

This is a textbook, not a research monograph, although I have attempted to provide enough detail to help researchers obtain the background necessary for exploration of more specialized literature. After teaching this material for over 40 years I hope that I have found helpful ways to make the subject understandable to students. I have certainly benefited greatly from their feedback, which has contributed to the changes in this second edition.

Although a comprehensive understanding of the laser and associated phenomena requires a quantum-mechanical treatment, almost all aspects of laser operation and important related phenomena can be explained well classically. Therefore, this text requires no knowledge of quantum mechanics, although a background in electromagnetic theory at the undergraduate level is desirable for a greater understanding. Most electrical engineering majors do not take a course in quantum mechanics until they reach graduate level, and many physics majors will not have acquired sufficient quantum mechanics knowledge at

the undergraduate level for this to make a meaningful contribution to better understanding in a study of lasers and electro-optics.

For all the above reasons this text should be suitable for senior undergraduate and beginning graduate students in electrical engineering or physics. It should also prove useful to other engineers or chemists who use lasers and electro-optic devices.

The text is broken up into two principal parts. Chapters 1–12 discuss the basic physics and engineering of lasers of all kinds, beginning with a discussion of the fundamental physics of the stimulated emission process and laser amplifiers. This is followed by chapters on laser resonators and the characteristics of laser radiation and methods for controlling it. There are succeeding chapters that cover optically pumped insulating crystal lasers, atomic gas lasers, and molecular gas lasers of various kinds including gas transport, gas dynamics and chemically pumped varieties, and tunable lasers. The first section of the book concludes with a chapter on semiconductor lasers that begins with a review of the basic physics necessary for their understanding.

The second part of the text covers various issues of relevance to lasers and electro-optics, including optical analysis and design techniques, the optics of Gaussian beams, laser resonators, and anisotropic crystalline materials. There are chapters on optical fibers, electro-optic and acousto-optic devices, the fundamentals of nonlinear optics, and application of nonlinear optics in harmonic generation, parametric processes, phase conjugation, and optical bistability. The text concludes with chapters on optical detectors and the detection process, coherence theory, and applications of lasers.

I have found that Chapters 1–12 provide sufficient material for a one semester course on lasers, with some applications from Chapter 23 included. Chapters 5 and 13–18, together with Chapter 22, form the basis of a one-semester course on optical design, electro-optic devices, and optical detectors. I draw on the somewhat more difficult material in Chapters 19–21 as reference material in both the one-semester courses just mentioned, and also as adjunct material in more advanced graduate courses.

I have been an experimentalist in the optics and laser business for over four decades. I have always found the laser itself a fascinating device that provides a teaching vehicle for discussing many fundamental physics concepts as well as practical aspects of optical design. I have always tried to introduce practical details of real lasers into my classes as early as a treatment of some of the associated fundamentals permits. This should be apparent in the current text where I digress in Chapter 3 into a fairly detailed practical discussion of two historically important lasers, even though contextually a fuller discussion of these devices could be left until later. I believe that this makes pedagogical sense as students get a glimpse of where they are headed. Throughout the text I have attempted to provide full details of important derivations, and provide practical examples from the literature on the design, construction, and performance characteristics of lasers and electro-optic devices.

In developing a sound approach to teaching the material in this text to many students over more than four decades I have drawn inevitably on the work of many others. There have been many other books that cover material that is shared in common with the current one. What is different between this and related texts is not so much the analytic treatment of common subject matter, but the specific choice of material presentation sequence, and

assorted explanations. As it is said, “there is nothing new under the sun,” so it is not the intent of this author to claim that the treatment of particular topics in the current text is necessarily unique. Different authors impart their own slant to the same subject matter: sometimes their treatments converge, particularly when there is one especially good way of explanation that is valid. I have attempted in every case to provide references to the original literature from which I have benefited in my writing, and I apologize for any inadvertent omissions. There is a vast literature on the subjects covered and would be it impossible to list all the important published work that has added to our knowledge. I have attempted to list key early work in each area covered, since use of the *Science Citations Index*¹ will in this way provide links to almost all the work in an area.

Over the course of many years I have learned much from my contacts in the classroom, office, and research laboratory, and at conferences. I am indebted to numerous past and present colleagues and students for their intellectual stimulation, collaboration in research, advice, provision of material, and feedback on early versions of the current text that have contributed greatly to the finished product. I would like to thank especially my past and present faculty colleagues Quirino Balzano, Kyuman Cho, Mario Dagenais, Julius Goldhar, Ping-Tong Ho, the late Urs Hochuli, Terry King, Chi Lee, Ross McFarlane, Stuart Milner, Ian Smith, Igor Smolyaninov; and finally George Wolga who gave me valuable help at the very beginning of this work. I am especially grateful to Professor Thomas Murphy for his helpful comments on how to treat the topic of semiconductor lasers, which just on its own could fill many volumes. My graduate students and post-doctoral research associates have over several years provided help and advice that have helped me greatly. I am particularly grateful to Navik Agrawal, David Coleman, Mohammed Eslami, Ehren Hwang, Jonathan Ko, Jaime Llorca, Billy Nelson, John Rzasa, Tommy Shen, and Chensheng Wu, for their current help with my research that has provided the time to finish this new edition. I am most grateful to Joan Hamilton and Nono Kusuma for drawing many of the diagrams and to Dave Mazzoni and Sarah Fish for help with additional diagrams.

I appreciate the patience of several editors at Cambridge University Press in waiting for completion of this second edition, and for accepting ongoing excuses as to why it was not completed earlier. I am most grateful to Patricia Keehn for her expert and careful computer typesetting work using $\text{\TeX}$ over enormous numbers of revisions of the first edition.²

Most of all, I am indebted to my family, especially my wife Mary, for their unfailing love and support over the course of my career.

Christopher C. Davis

College Park, Maryland
November 2013

¹ Part of the *Web of Science*, a subscription data base that is widely available: <http://thomsonreuters.com/web-of-science/>
² In most cases where curves showing the parametric variation of phenomena discussed in the text are given these have been calculated from scratch using Mathcad (©Mathsoft, Inc.) The entire text has been typeset using Donald Knuth’s $\text{\TeX}$: *The $\text{\TeX}$ book*, Addison-Wesley, Reading, MA, 1984. $\text{\TeX}$ is a trademark of the American Mathematical Society.