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Lectures on the Combinatorics of Free Probability

Alexandru Nica
University of Waterloo, Ontario

Roland Speicher
Queen's University, Ontario

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Introduction

Free probability theory is a quite recent theory, bringing together many different fields of mathematics, for example operator algebras, random matrices, combinatorics, or representation theory of symmetric groups. So it has a lot to offer to various mathematical communities, and interest in free probability has steadily increased in recent years.

However, this diversity of the field also has the consequence that it is considered hard to access for a beginner. Most of the literature on free probability consists of a mixture of operator algebraic and probabilistic notions and arguments, interwoven with random matrices and combinatorics.

Whereas more advanced operator algebraic or probabilistic expertise might indeed be necessary for a deeper appreciation of special applications in the respective fields, the basic core of the theory, however, can be mostly freed from this and it is possible to give a fairly elementary introduction to the main notions, ideas and problems of free probability theory. The present lectures are intended to provide such an introduction.

Our main emphasis will be on the combinatorial side of free probability. Even when stripped from analytical structure, the main features of free independence are still present; moreover, even on this more combinatorial level it is important to organize all relevant information about the considered variables in the right way. Anyone who has tried to perform computations of joint distributions for non-commuting variables will probably agree that they tend to be horribly messy if done in a naive way. One of the main goals of the book is to show how such computations can be vastly simplified by appropriately relying on a suitable combinatorial structure – the lattices of non-crossing partitions. The combinatorial development starts from the standard theory of Möbius inversion on non-crossing partitions, but has its own specific flavor – one arrives to a theory of free or non-crossing cumulants or, in an alternative approach, one talks about R-transforms for non-commutative random variables.

While writing this book, there were two kinds of readers that we had primarily in mind:

(a) a reader with background in operator algebras or probability who wants to see the more advanced "tools of the trade" on the combinatorial side of free probability;
(b) a reader with background from algebraic combinatorics who wants to get acquainted with a field (and a possible source of interesting problems) where non-trivial combinatorial tools are used.

We wrote our lectures by trying to accommodate readers from both these categories. The result is a fairly elementary exposition, which should be accessible to a beginning graduate student or even to a strong senior undergraduate student.

Free probability also has applications outside of mathematics, in particular in electrical engineering. Our exposition should also be useful for readers with engineering background, who have seen the use of R- or S-transform techniques in applications, for example in wireless communications, and who want to learn more about the underlying theory.

We emphasize that the presentation style used throughout the book is a detailed one, making the material largely self-contained, and only rarely requiring that other textbooks or research papers are consulted. The basic units of this book are called "lectures." They were written following the idea that the material contained in one of them should be suitable for being presented in one class of a first-year graduate course. (We have in mind a class of 90 minutes, where the instructor presents the essential points of the lecture, and leaves a number of things for individual study.)

While the emphasis is on combinatorial aspects, we still felt that we must give an introduction of how the general framework of free probability comes about. Also, we felt that the flavor of the theory will be better conveyed if we show, with moderation and within a self-contained exposition, how analytical arguments can be interwoven with the combinatorial ones. However, it should be understood that in the analytical respects, this book is only an appetizer and an invitation to further reading. In particular, the analytical framework used for illustrations is exclusively that of a C^*-probability space. The reader should be aware that some of the most significant applications of free probability to operator algebras take place in the more elaborate framework of W^*-probability spaces; but going to W^*-structures (or in other words, to von Neumann algebra theory) did not seem possible within the detailed, self-contained style of the book, and within the given page limits.

A consequence of the frugality of the analytic aspects covered by the book is that we do not discuss free entropy and free Fisher information, and how free cumulants can be used in some cases to perform free information calculations. Free entropy is currently one of the main directions of development in free probability; for an overview of the topic see the recent survey by Voiculescu [**85**].

Coming to things that are not covered by the book we must also say, with regret, that we only consider free independence over the complex field. The combinatorial ideas of free probability have a far-reaching extension to the situation when free independence is considered over an algebra $\mathcal{B}$ (instead of just $\mathbb{C}$) – the reader interested in this direction is referred to the memoir [**73**].

References to the literature are not made in the body of the lectures, but are collected in the "Notes and comments" section at the end of the book. The literature on free probability is growing at an explosive rate, and, with due apologies, we felt it is beyond our limits to even try to provide an exhaustive bibliography. We have followed the line of only citing the research work which is presented in the lectures, or is very directly connected to it. For a more complete image of work in this field, the reader can consult the survey papers indicated at the beginning of the "Notes and comments" section.

So, to summarize, from one point of view this is a research monograph, presenting the current state of the combinatorial facet of free probability. At the same time it is an introduction to the field – one which is, we hope, friendly and self-contained. Finally, the book is written with the specific purpose of being used for teaching a course. We hope this will be a contribution towards making free probability appear more often as a topic for a graduate course, and we look forward to hearing from other people how following these lectures has worked for them.

Finally we would like to mention that the idea of writing this book came from a sequence of lectures which we gave at the Henri Poincaré Institute in Paris, during a special semester on free probability and operator spaces hosted by the institute in Fall 1999. Time has flown quickly since then, but we hope it is not too late to thank the Poincaré Institute, and particularly the organizers of that special semester – Philippe Biane, Gilles Pisier, and Dan Voiculescu – for the great environment they offered us, and for the opportunity of getting started on this project.