Cambridge University Press
978-0-521-85811-3 - Dynamics of Particles and Rigid Bodies: A Systematic Approach
Anil V. Rao
Excerpt
More information

Chapter 1

Introductory Concepts

The scientist does not study nature because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful. If nature were not beautiful, it would not be worth knowing, and if nature were not worth knowing, life would not be worth living.

- Jules Henri Poincaré (1854–1912)
French Mathematician and Physicist

Mechanics is the study of the effect that physical forces have on objects. *Dynamics* is the particular branch of mechanics that deals with the study of the effect that forces have on the motion of objects. Dynamics is itself divided into two branches called Newtonian dynamics and relativistic dynamics. *Newtonian* dynamics is the study of the motion of objects that travel with speeds significantly less than the speed of light while *relativistic* dynamics is the study of the motion of objects that travel with speeds at or near the speed of light. This division in the subject of dynamics arises because the physics associated with the motion of objects that travel with speeds much less than the speed of light can be modeled much more simply than the physics associated with the motion of objects that travel with speeds at or near the speed of light. Moreover, nonrelativistic dynamics deals primarily with the motion of objects on a macroscopic scale while relativistic dynamics deals with the study of the motion of objects on a microscopic or submicroscopic scale. The objective of this book is to present the underlying concepts of Newtonian dynamics in a clear and concise manner and to develop a systematic framework for solving problems in classical Newtonian dynamics.

As with any subject that is based on the laws of physics, Newtonian dynamics needs to be described using mathematics. More specifically, it must be possible to describe the physical laws in a way that is independent of the particular coordinate system in which one chooses to formulate a particular problem. The mathematical approach that gives us the freedom to develop a coordinate-free approach to Newtonian mechanics is that of *vector and tensor algebra*.

Once the physical laws have been described in a coordinate-free manner, the next step is to formulate the particular problem of interest. While the basic laws themselves are coordinate-free, to solve a particular problem it is necessary to specify all relevant quantities using a coordinate system of choice. While in principle it is possible to use any coordinate system to describe the motion of a material body, choosing a particular coordinate system could vastly simplify the particular problem under consideration. The remainder of this chapter is devoted to providing a review of the vector and tensor algebra required to formulate and analyze problems in nonrelativistic mechanics. While this chapter provides a mathematical overview, it is not intended as a substitute for a book on engineering mathematics. For a more in-depth presentation of engineering mathematics, the reader is referred to a standard text in undergraduate engineering mathematics such as that found in Kreyszig (1988).

1.1 Scalars

A *scalar* is any quantity that is expressible as a real number. We denote a scalar by a non-boldface character and denote the set of real numbers by $\mathbb{R}$, i.e., we say that the (non-boldface) quantity a is a scalar if

$$a \in \mathbb{R}$$

Scalars satisfy the following properties with respect to addition and multiplication:

1. Commutativity: For all $a \in \mathbb{R}$ and $b \in \mathbb{R}$,

$$\begin{aligned} a + b &= b + a \\ ab &= ba \end{aligned}$$

2. Associativity: For all $a \in \mathbb{R}$, $b \in \mathbb{R}$, and $c \in \mathbb{R}$,

$$\begin{aligned} (a + b) + c &= a + (b + c) \\ a(bc) &= (ab)c \end{aligned}$$

3. Zero Scalar: There exists a scalar 0 such that for all $a \in \mathbb{R}$,

$$\begin{aligned} a + 0 &= 0 + a = a \\ 0(a) &= (a)0 = 0 \end{aligned}$$

4. Unit Scalar: there exists a scalar 1 such that for all $a \in \mathbb{R}$,

$$1(a) = (a)1 = a$$

5. Inverse scalar: For all $a \neq 0 \in \mathbb{R}$, there exists a scalar $1/a$ such that

$$\frac{1}{a}(a) = a\frac{1}{a} = 1$$

6. Negativity: There exists a scalar -1 such that for all $a \in \mathbb{R}$

$$\begin{aligned} -1(a) &= a(-1) = -a \\ a + (-a) &= (-a) + a = 0 \end{aligned}$$

1.2 Vectors

A *vector* is any quantity that has both *magnitude and direction*. A vector is denoted by a *boldface* character, i.e., a quantity $\mathbf{a}$ is a vector. Because the study of Newtonian mechanics focuses on the motion of objects in *three-dimensional Euclidean space*, throughout this book we will be interested in three-dimensional vectors. Three-dimensional Euclidean space is denoted $\mathbb{R}^3$. Consequently, the notation

$$\mathbf{a} \in \mathbb{R}^3$$

means that the vector $\mathbf{a}$ lies in $\mathbb{R}^3$.

The *length* of a vector $\mathbf{a} \in \mathbb{R}^3$ is called the *magnitude* of $\mathbf{a}$. The magnitude or *Euclidean norm* of a vector $\mathbf{a}$ is denoted $\|\mathbf{a}\|$ and is a scalar, i.e., $\|\mathbf{a}\| \in \mathbb{R}$. A vector whose magnitude is zero is called the *zero vector*. We denote the zero vector by a boldface zero, i.e., the zero vector is denoted by $\mathbf{0}$. The *direction* of a nonzero vector $\mathbf{a}$ is the vector divided by its magnitude, i.e., the direction of the vector $\mathbf{a}$, denoted $\mathbf{u}_a$, is given as

$$\mathbf{u}_a = \frac{\mathbf{a}}{\|\mathbf{a}\|}$$

Furthermore, the direction of a nonzero vector is called a *unit vector* because its magnitude is unity, i.e., $\|\mathbf{u}_a\| = 1$. Two vectors are said to be equal if they have the same magnitude and direction.

1.2.1 Types of Vectors

While geometrically a vector is any quantity with magnitude and direction, the physical effect of a vector $\mathbf{a}$ on a mechanical system may depend in addition on a particular line of action in $\mathbb{R}^3$ or a particular point in $\mathbb{R}^3$. In particular, vectors arising in mechanics fall into one of three categories[1]: (a) free vectors; (b) sliding vectors; and (c) bound vectors. Each type of vector is now described in more detail.

A *free vector* is any vector $\mathbf{b}$ with no specified line of action or point of application in $\mathbb{R}^3$. Figure 1–1 shows an example of two identical free vectors $\mathbf{b}$ and $\mathbf{b}'$. While $\mathbf{b}$ and $\mathbf{b}'$ have the same direction and magnitude, they do not share the same start or end point. In particular, $\mathbf{b}$ starts at point Q and ends at point P while $\mathbf{b}'$ starts at point $Q' \neq Q$ and ends at point $P' \neq P$. However, because $\mathbf{b}$ and $\mathbf{b}'$ have the same direction and magnitude, they are identical free vectors. Examples of free vectors are the angular velocity of a reference frame or a rigid body, a pure torque applied to a rigid body, and a basis vector.

A *sliding vector* is any vector $\mathbf{b}$ that has a specified *line of action* or *axis* in $\mathbb{R}^3$, but has no specified point of application in $\mathbb{R}^3$. Figure 1–2 shows two identical sliding vectors $\mathbf{b}$ and $\mathbf{b}'$. As with free vectors, $\mathbf{b}$ and $\mathbf{b}'$ have the same magnitude and direction. However, while the vector $\mathbf{b}$ starts at point Q and ends at point P, the vector $\mathbf{b}'$ starts at point $Q' \neq Q$ and ends at point $P' \neq P$ (where the points P, Q, P', and Q' are colinear). Consequently, $\mathbf{b}$ and $\mathbf{b}'$ are identical sliding vectors, but are different free vectors. An example of a sliding vector is the force applied to a rigid body.

A *bound vector* is any vector that has both a specified line of action in $\mathbb{R}^3$ and a specified point of application in $\mathbb{R}^3$. From its definition, it can be seen that a bound

[1]An excellent description of free, sliding, and bound vectors can be found in either Synge and Griffith (1959) or Greenwood (1988).

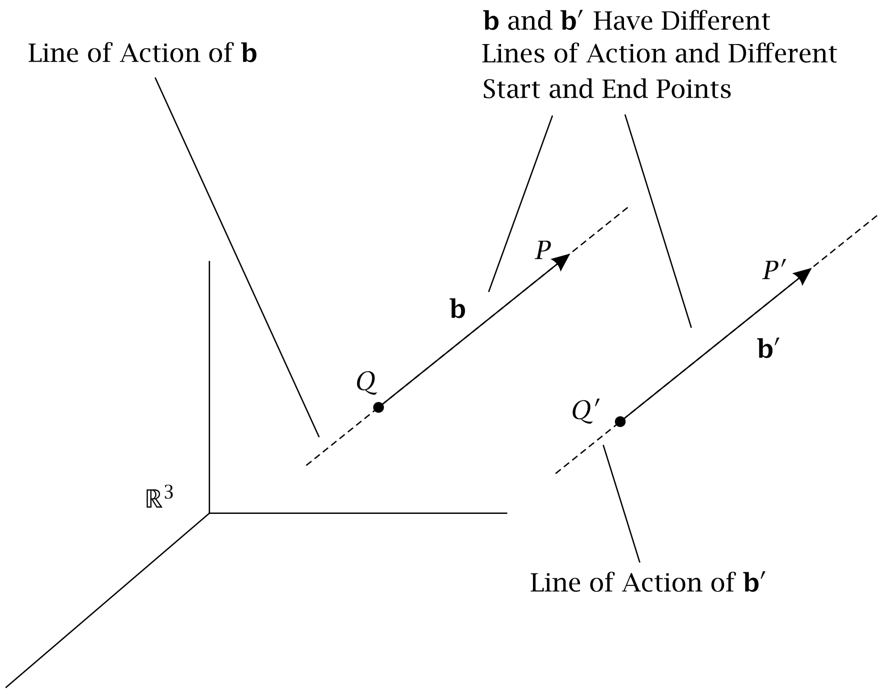

Figure 1–1 Two equal free vectors **b** and **b**′ that have the same direction and magnitude, but different lines of action and different start and end points.

vector is unique, i.e., only *one* vector can have a specified direction, magnitude, line of action, and origin. An example of a bound vector is the force acting on or exerted by an elastic body (e.g., the force exerted by a linear spring); in the case of an elastic body, the deformation of the body depends on the changing point of application of the force.

It should be noted that vector algebra is valid only for free vectors. However, because all vectors are defined by their direction and magnitude, vector algebra can be performed on sliding and bound vectors by treating them *as though* they are free vectors. Consequently, the result of any algebraic operation on vectors, regardless of the type of vector, results in a free vector. From this point forth, unless otherwise stated or additional clarification is necessary, all vectors will be assumed to be free vectors.

1.2.2 Addition of Vectors

Let $\mathbf{a}$ and $\mathbf{b}$ be vectors in $\mathbb{R}^3$. Then the *sum* of $\mathbf{a}$ and $\mathbf{b}$, denoted $\mathbf{c}$, is given as

$$\mathbf{c} = \mathbf{a} + \mathbf{b} \tag{1–1}$$

Vector addition has the following properties:

1. Commutativity: For all $\mathbf{a} \in \mathbb{R}^3$ and $\mathbf{b} \in \mathbb{R}^3$,

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$

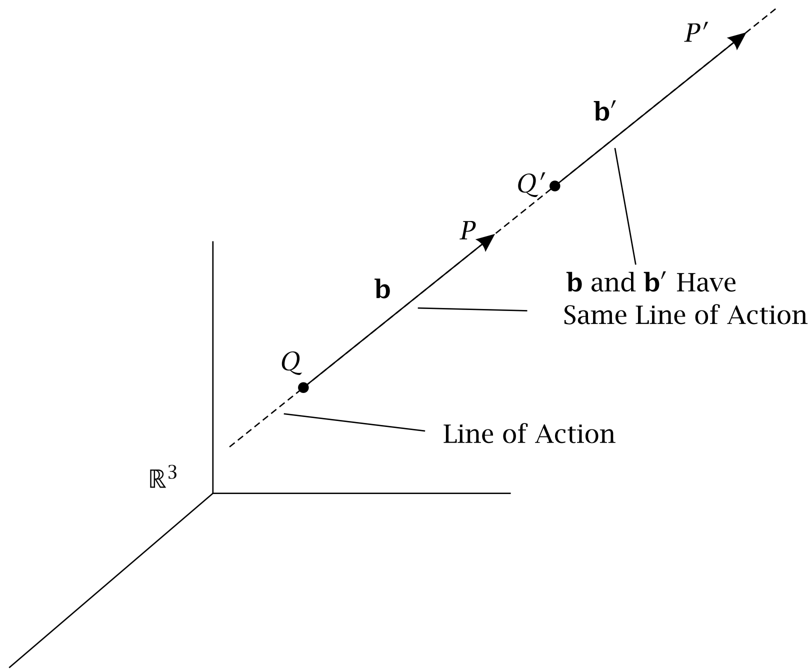

Figure 1–2 Two equal sliding vectors $\mathbf{b}$ and $\mathbf{b}'$ that have the same direction, magnitude, and line of action, but different start and end points.

2. Associativity: For all $\mathbf{a} \in \mathbb{R}^3$, $\mathbf{b} \in \mathbb{R}^3$, and $\mathbf{c} \in \mathbb{R}^3$,

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$$

3. Zero vector: There exists a vector $\mathbf{0}$ such that for all $\mathbf{a} \in \mathbb{R}^3$,

$$\mathbf{a} + \mathbf{0} = \mathbf{a}$$

4. For all $\mathbf{a} \in \mathbb{R}^3$, there exists $-\mathbf{a} \in \mathbb{R}^3$ such that

$$\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$$

1.2.3 Components of a Vector

Any vector $\mathbf{a} \in \mathbb{R}^3$ can be expressed in terms of three noncoplanar vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ called *basis vectors*. Correspondingly, any noncoplanar set of vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is called a *basis* for $\mathbb{R}^3$. In terms of the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, the vector $\mathbf{a}$ can be written as

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \tag{1–2}$$

where a_1, a_2, and a_3 are the *components* of $\mathbf{a}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Generally speaking, it is preferable to use a basis of mutually orthogonal vectors. Any basis consisting of mutually orthogonal vectors is called an *orthogonal basis*. Even more specifically, it is most preferable to use a basis consisting of mutually orthogonal *unit* vectors. A basis consisting of mutually orthogonal unit vectors is called an *orthonormal basis*. In the remainder of this book, we will restrict our attention to orthonormal bases. To this end, we will use the term "basis" to mean specifically an orthonormal basis. The representation of a vector **a** in an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is shown schematically in Fig. 1–3. Using the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, we can resolve two vectors **a** and **b** into

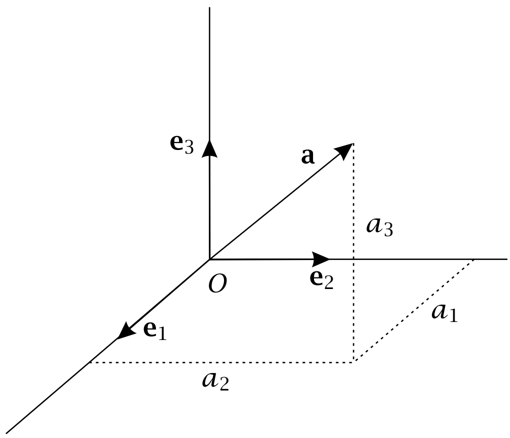

Figure 1–3 Vector **a** expressed in an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as follows:

$$\begin{aligned} \mathbf{a} &= a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \\ \mathbf{b} &= b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 \end{aligned} \tag{1–3}$$

Then the sum of **a** and **b** is given in terms of $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as

$$\mathbf{c} = (a_1 + b_1)\mathbf{e}_1 + (a_2 + b_2)\mathbf{e}_2 + (a_3 + b_3)\mathbf{e}_3 \tag{1–4}$$

1.2.4 Multiplication of a Vector by a Scalar

Let **a** be a vector in $\mathbb{R}^3$ and let $k \in \mathbb{R}$ be a scalar. Then the product of **a** with the scalar k, denoted $k\mathbf{a}$, has the following properties:

1. $\|k\mathbf{a}\| = |k|\|\mathbf{a}\|$
2. $\dfrac{k\mathbf{a}}{\|k\mathbf{a}\|} = \dfrac{\mathbf{a}}{\|\mathbf{a}\|}$ if $k > 0$ and $\mathbf{a} \neq \mathbf{0}$
3. $\dfrac{k\mathbf{a}}{\|k\mathbf{a}\|} = -\dfrac{\mathbf{a}}{\|\mathbf{a}\|}$ if $k < 0$ and $\mathbf{a} \neq \mathbf{0}$
4. $k\mathbf{a} = \mathbf{0}$ if either $\mathbf{a} = \mathbf{0}$ or $k = 0$
5. $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$
6. $(k_1 + k_2)\mathbf{a} = k_1\mathbf{a} + k_2\mathbf{a}$
7. $k_2(k_1\mathbf{a}) = k_2k_1\mathbf{a}$
8. $(1)\mathbf{a} = \mathbf{a}(1) = \mathbf{a}$
9. $(0)\mathbf{a} = \mathbf{a}(0) = \mathbf{0}$

10. $(-1)\mathbf{a} = \mathbf{a}(-1) = -\mathbf{a}$

Finally, if $\mathbf{a}$ is expressed in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, then $k\mathbf{a}$ is given as

$$k\mathbf{a} = ka_1\mathbf{e}_1 + ka_2\mathbf{e}_2 + ka_3\mathbf{e}_3 \tag{1–5}$$

1.2.5 Scalar Product

Let $\mathbf{a}$ and $\mathbf{b}$ be vectors in $\mathbb{R}^3$. Then the *scalar product* or *dot product* between $\mathbf{a}$ and $\mathbf{b}$ is defined as

$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\|\cos\theta = ab\cos\theta \tag{1–6}$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. The scalar product has the following properties:

1. $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$
2. $\mathbf{a}\cdot(k\mathbf{b}) = k\mathbf{a}\cdot\mathbf{b}$ where $k \in \mathbb{R}$
3. $(\mathbf{a}+\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{c}$

Two nonzero vectors are said to be *orthogonal* if their scalar product is *zero*, i.e., $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if

$$\mathbf{a}\cdot\mathbf{b} = 0 \quad (\mathbf{a},\mathbf{b} \neq \mathbf{0}) \tag{1–7}$$

A set of vectors $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is said to be *mutually orthogonal* if

$$\mathbf{a}_i\cdot\mathbf{a}_j = 0 \quad (i \neq j,\ i, j = 1, \ldots, n) \tag{1–8}$$

Finally, the magnitude of a vector $\mathbf{a}$ is equal to the square root of the dot product of the vector with itself, i.e.,

$$\|\mathbf{a}\| = \sqrt{\mathbf{a}\cdot\mathbf{a}} \tag{1–9}$$

Suppose now that $\mathbf{a}$ and $\mathbf{b}$ are expressed in a particular basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as

$$\begin{aligned} \mathbf{a} &= a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \\ \mathbf{b} &= b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 \end{aligned} \tag{1–10}$$

Then the scalar product of $\mathbf{a}$ with $\mathbf{b}$ is given as

$$\mathbf{a}\cdot\mathbf{b} = (a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3)\cdot(b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3) \tag{1–11}$$

Because we are restricting attention to orthonormal bases, the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ satisfies the properties that

$$\mathbf{e}_i\cdot\mathbf{e}_j = \begin{cases} 1 \ (i = j) \\ 0 \ (i \neq j) \end{cases} \quad (i, j = 1, 2, 3) \tag{1–12}$$

Consequently, we have

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 \tag{1–13}$$

Using Eq. (1–13) and the definition of the magnitude of a vector as given in Eq. (1–9), the magnitude of a vector $\mathbf{a}$ can be written in terms of the components of $\mathbf{a}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2} \tag{1–14}$$

1.2.6 Vector Product

Let $\mathbf{a}$ and $\mathbf{b}$ be vectors in $\mathbb{R}^3$. Then the *vector product* or *cross product* between two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\| \sin\theta\,\mathbf{n} \tag{1–15}$$

where $\mathbf{n}$ is the unit vector in the direction orthogonal to both $\mathbf{a}$ and $\mathbf{b}$ in a right-handed sense and θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. The term “right-handed sense” arises from the fact that the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ assume an orientation that corresponds to the index finger, middle finger, and thumb of the right hand when these three fingers are held as shown in Fig. 1–4.

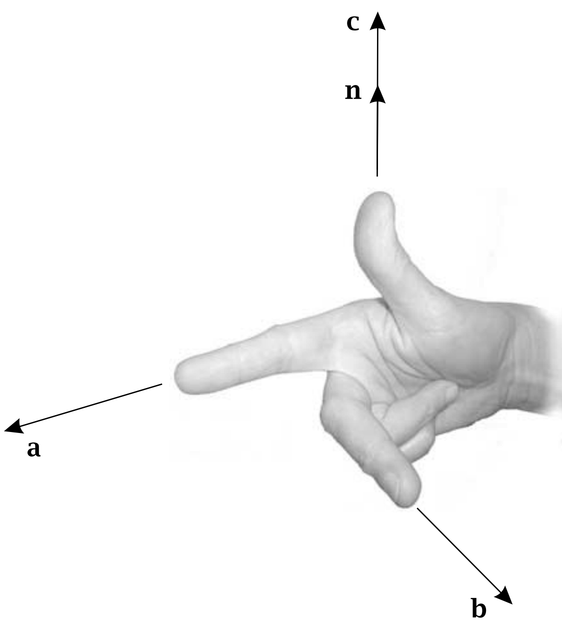

Figure 1–4 Schematic of right-hand rule corresponding to the vector product of two vectors using the index finger, middle finger, and thumb of the right hand.

The magnitude of the vector product of two vectors is given as

$$\|\mathbf{c}\| = \|\mathbf{a}\|\,\|\mathbf{b}\| \sin\theta \tag{1–16}$$

The vector product has the following properties:

1. $\mathbf{a} \times \mathbf{a} = \mathbf{0}$
2. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
3. $(k\mathbf{a}) \times \mathbf{b} = k(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (k\mathbf{b})$ where $k \in \mathbb{R}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c})$

Now suppose that $\mathbf{a}$ and $\mathbf{b}$ are expressed in terms of the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, i.e.,

$$\begin{aligned} \mathbf{a} &= a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \\ \mathbf{b} &= b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 \end{aligned} \tag{1–17}$$

Then the cross product of **a** and **b** is given as

$$\mathbf{a} \times \mathbf{b} = (a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) \times (b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3) \tag{1–18}$$

Expanding Eq. (1–18), we obtain

$$\begin{aligned} \mathbf{a} \times \mathbf{b} = {} & a_1b_2\mathbf{e}_1 \times \mathbf{e}_2 + a_1b_3\mathbf{e}_1 \times \mathbf{e}_3 + a_2b_1\mathbf{e}_2 \times \mathbf{e}_1 \\ & + a_2b_3\mathbf{e}_2 \times \mathbf{e}_3 + a_3b_1\mathbf{e}_3 \times \mathbf{e}_1 + a_3b_2\mathbf{e}_3 \times \mathbf{e}_2 \end{aligned} \tag{1–19}$$

Again we remind the reader that we are restricting our attention to orthonormal bases. Furthermore, suppose that the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ forms a right-handed set, i.e., $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ satisfies the following properties:

$$\begin{aligned} \mathbf{e}_1 \times \mathbf{e}_2 &= \mathbf{e}_3 \\ \mathbf{e}_2 \times \mathbf{e}_3 &= \mathbf{e}_1 \\ \mathbf{e}_3 \times \mathbf{e}_1 &= \mathbf{e}_2 \end{aligned} \tag{1–20}$$

Then $\mathbf{a} \times \mathbf{b}$ is given as

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{e}_1 + (a_3b_1 - a_1b_3)\mathbf{e}_2 + (a_1b_2 - a_2b_1)\mathbf{e}_3 \tag{1–21}$$

In terms of a right-handed basis, Eq. (1–21) can also be written as the following determinant (Kreyszig, 1988):

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{1–22}$$

1.2.7 Scalar Triple Product

Given three vectors **a**, **b**, and **c**, the *scalar triple product* is defined as

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \tag{1–23}$$

The scalar triple product has the following properties:

1. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$
2. $\mathbf{a} \cdot (k\mathbf{b} \times \mathbf{c}) = k\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$

Suppose that the vectors **a**, **b**, and **c** are each expressed in an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as

$$\begin{aligned} \mathbf{a} &= a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \\ \mathbf{b} &= b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 \\ \mathbf{c} &= c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3 \end{aligned} \tag{1–24}$$

Then the scalar triple product can be written as

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) \tag{1–25}$$

The scalar triple product can also be written as the following determinant:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{1–26}$$

Finally, the scalar triple product can be written as

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \|\mathbf{a}\| \|\mathbf{b} \times \mathbf{c}\| \cos\theta \tag{1–27}$$

where θ is the angle between the vector **a** and the vector $\mathbf{b} \times \mathbf{c}$.

1.2.8 Vector Triple Product

Given three vectors **a**, **b**, and **c**, the *vector triple product* is given as

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \tag{1–28}$$

The vector triple product can be written as

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \tag{1–29}$$

Suppose that the vectors **a**, **b**, and **c** are each expressed in an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as

$$\begin{array}{rcl} \mathbf{a} & = & a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \\ \mathbf{b} & = & b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 \\ \mathbf{c} & = & c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3 \end{array} \tag{1–30}$$

The vector triple product can then be written as

$$\begin{array}{rl} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = & (a_1c_1 + a_2c_2 + a_3c_3)(b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3) \\ & - (a_1b_1 + a_2b_2 + a_3b_3)(c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3) \end{array} \tag{1–31}$$

1.3 Tensors

A *tensor* (or second-order tensor[2]), denoted **T**, is a linear operator that associates a vector $\mathbf{a} \in \mathbb{R}^3$ to another vector $\mathbf{b} \in \mathbb{R}^3$, i.e., if **T** is a tensor and $\mathbf{a} \in \mathbb{R}^3$ is a vector, then there exists a vector **b** such that

$$\mathbf{b} = \mathbf{T} \cdot \mathbf{a} \tag{1–32}$$

It is noted that the binary operator "·" in Eq. (1–32) is different from the scalar product between two vectors in that the "·" denotes the operation of the tensor **T** on the vector **a**. Now, because tensors are linear operators, they satisfy the following properties:

1. For all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$,
$$\mathbf{T} \cdot (\mathbf{a} + \mathbf{b}) = \mathbf{T} \cdot \mathbf{a} + \mathbf{T} \cdot \mathbf{b}$$
2. For all $\mathbf{a} \in \mathbb{R}^3$ and $k \in \mathbb{R}$,
$$\mathbf{T} \cdot (k\mathbf{a}) = k\mathbf{T} \cdot \mathbf{a}$$
3. There exists a *zero tensor*, denoted **O**, such that for every $\mathbf{a} \in \mathbb{R}^3$,
$$\mathbf{O} \cdot \mathbf{a} = \mathbf{0} \tag{1–33}$$
where **0** is the zero vector.
4. There exists an *identity tensor* or *unit tensor*, denoted **U**, such that for every $\mathbf{a} \in \mathbb{R}^3$,
$$\mathbf{U} \cdot \mathbf{a} = \mathbf{a} \tag{1–34}$$

[2]Strictly speaking, the tensor defined in Eq. (1–32) is a *second-order* tensor. While tensor algebra generalizes well beyond second-order tensors, in this book we will only be concerned with second-order tensors. Consequently, throughout this book we will use the term "tensor" to mean "second-order tensor".