Cambridge University Press
978-0-521-85784-0 - An Introduction to Godel's Theorems
Peter Smith
Frontmatter
More information

An Introduction to Gödel's Theorems

In 1931, the young Kurt Gödel published his First Incompleteness Theorem, which tells us that, for any sufficiently rich theory of arithmetic, there are some arithmetical truths the theory cannot prove. This remarkable result is among the most intriguing (and most misunderstood) in logic. Gödel also outlined an equally significant Second Incompleteness Theorem. How are these Theorems established, and why do they matter? Peter Smith answers these questions by presenting an unusual variety of proofs for the First Theorem, showing how to prove the Second Theorem, and exploring a family of related results (including some not easily available elsewhere). The formal explanations are interwoven with discussions of the wider significance of the two Theorems. This book will be accessible to philosophy students with a limited formal background. It is equally suitable for mathematics students taking a first course in mathematical logic.

Peter Smith is Lecturer in Philosophy at the University of Cambridge. His books include *Explaining Chaos* (1998) and *An Introduction to Formal Logic* (2003), and he is a former editor of the journal of *Analysis*.

Cambridge University Press
978-0-521-85784-0 - An Introduction to Godel's Theorems
Peter Smith
Frontmatter
More information

An Introduction to Gödel's Theorems

Peter Smith
University of Cambridge

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521674539

© Peter Smith 2007

This publication is in copyright. Subject to statutory exception and
to the provisions of relevant collective licensing agreements,
no reproduction of any part may take place without the written
permission of Cambridge University Press.

First published 2007

Printed in the United Kingdom at the University Press, Cambridge

Typeset by the author in Computer Modern 10/12.5 pt: system LaTeX 2_ε

A catalogue record for this publication is available from the British Library

ISBN 978-0-521-85784-0 hardback
ISBN 978-0-521-67453-9 paperback

Cambridge University Press has no responsibility for the persistence
or accuracy of URLs for external or third-party internet websites
referred to in this publication, and does not guarantee that any
content on such websites is, or will remain, accurate or appropriate.

CAMBRIDGE

Cambridge University Press
978-0-521-85784-0 - An Introduction to Godel's Theorems
Peter Smith
Frontmatter
More information

For Patsy, as ever

© Cambridge University Press www.cambridge.org

Contents

© Cambridge University Press www.cambridge.org

© Cambridge University Press www.cambridge.org

Cambridge

Cambridge University Press
978-0-521-85784-0 - An Introduction to Godel's Theorems
Peter Smith
Frontmatter
More information

© Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-0-521-85784-0 - An Introduction to Godel's Theorems
Peter Smith
Frontmatter
More information

© Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-0-521-85784-0 - An Introduction to Godel's Theorems
Peter Smith
Frontmatter
More information

© Cambridge University Press www.cambridge.org

Preface

In 1931, the young Kurt Gödel published his First and Second Incompleteness Theorems; very often, these are simply referred to as 'Gödel's Theorems'. His startling results settled (or at least, seemed to settle) some of the crucial questions of the day concerning the foundations of mathematics. They remain of the greatest significance for the philosophy of mathematics – though just what that significance is continues to be debated. It has also frequently been claimed that Gödel's Theorems have a much wider impact on very general issues about language, truth and the mind.

This book gives proofs of the Theorems and related formal results, and touches – necessarily briefly – on some of their implications. Who is this book for? Roughly speaking, for those who want a lot more fine detail than you get in books for a general audience (the best of those is Franzén, 2005), but who find the rather forbidding presentations in classic texts in mathematical logic (like Mendelson, 1997) too short on explanatory scene-setting. So I hope philosophy students taking an advanced logic course will find the book useful, as will mathematicians who want a more accessible exposition.

But don't be misled by the relatively relaxed style; don't try to browse through too quickly. We do cover a lot of ground in quite a bit of detail, and new ideas often come thick and fast. Take things slowly!

Theorems are numbered in the standard way, so Theorem 16.2 is the second theorem in Chapter 16. The distinction between a 'proof', a 'proof sketch' and (occasionally) a 'sketch of a proof sketch' is blurry. The end of a proof or proof sketch is marked by ⊠.

I assume only a modest amount of background in logic. So we can't cover, for example, material that presupposes a serious knowledge of model theory; which means that we don't discuss (say) model-theoretic arguments for incompleteness. That's a pity. But there's a sequel planned for enthusiasts who want to know about such matters.

I originally intended to write a rather shorter book, leaving many of the formal details to be filled in from elsewhere. But while that plan might have suited some readers, I soon realized that it would seriously irritate others to be sent hither and thither to consult a variety of textbooks with different terminologies and different notations. So in the end, I have given more or less full proofs of most of the key results we cover. However, my original plan shows through in two ways. First, some proofs are still only partially sketched in. Second, I try to signal very clearly when the detailed proofs I do give can be skipped without much loss of

© Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-0-521-85784-0 - An Introduction to Godel's Theorems
Peter Smith
Frontmatter
More information

understanding. With judicious skimming, you should be able to follow the main formal themes of the book even if you start from a very modest background in logic. For those who want to fill in more details and test their understanding there are exercises on the book's website at www.godelbook.net.

As we go through, there is also a small amount of broadly philosophical commentary. I follow Gödel in believing that our formal investigations and our general reflections on foundational matters should illuminate and guide each other. I hope that the more philosophical discussions – relatively elementary though certainly not always uncontentious – will also be reasonably widely accessible. Note however that I am more interested in patterns of ideas and arguments than in being historically very precise when talking e.g. about logicism or about Hilbert's Programme.

Writing a book like this presents many problems of organization. At various points we will need to call upon some background ideas from general logical theory. Do we explain them all at once, up front? Or do we introduce them as we go along, when needed? Similarly we will also need to call upon some ideas from the general theory of computation – for example, we will make use of both the notion of a 'primitive recursive function' and the more general notion of a 'μ-recursive function'. Again, do we explain these together? Or do we give the explanations many chapters apart, when the respective notions first get used?

I've mostly adopted the second policy, introducing new ideas as and when needed. This has its costs, but I think that there is a major compensating benefit, namely that the way the book is organized makes it clearer just what depends on what. It also reflects something of the historical order in which ideas emerged.

My colleague Michael Potter has been an inspiring presence since I returned to Cambridge. Many thanks are due to him and to all those who have very kindly given me comments on parts of various drafts, including the late Torkel Franzén, Tim Button, Luca Incurvati, Jeffrey Ketland, Aatu Koskensilta, Christopher Leary, Mary Leng, Toby Ord, Alex Paseau, Jacob Plotkin, José F. Ruiz, Kevin Scharp, Hartley Slater, and Tim Storer. I should especially mention Richard Zach, whose comments at two different stages were particularly extensive and particularly helpful.

But my greatest debt is to Patsy Wilson-Smith, without whose continuing love and support this book would never have been written.

© Cambridge University Press www.cambridge.org