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Peter Smith

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An Introduction to Gödel's Theorems

In 1931, the young Kurt Gödel published his First Incompleteness Theorem, which tells us that, for any sufficiently rich theory of arithmetic, there are some arithmetical truths the theory cannot prove. This remarkable result is among the most intriguing (and most misunderstood) in logic. Gödel also outlined an equally significant Second Incompleteness Theorem. How are these Theorems established, and why do they matter? Peter Smith answers these questions by presenting an unusual variety of proofs for the First Theorem, showing how to prove the Second Theorem, and exploring a family of related results (including some not easily available elsewhere). The formal explanations are interwoven with discussions of the wider significance of the two Theorems. This book will be accessible to philosophy students with a limited formal background. It is equally suitable for mathematics students taking a first course in mathematical logic.

PETER SMITH is Lecturer in Philosophy at the University of Cambridge. His books include *Explaining Chaos* (1998) and *An Introduction to Formal Logic* (2003), and he is a former editor of the journal of *Analysis*.

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[More information](#)

For Patsy, as ever

Contents

Preface	xiii
1 What Gödel's Theorems say	1
Basic arithmetic · Incompleteness · More incompleteness · Some implications? · The unprovability of consistency · More implications? · What's next?	
2 Decidability and enumerability	8
Functions · Effective decidability, effective computability · Enumerable sets · Effective enumerability · Effectively enumerating pairs of numbers	
3 Axiomatized formal theories	17
Formalization as an ideal · Formalized languages · Axiomatized formal theories · More definitions · The effective enumerability of theorems · Negation-complete theories are decidable	
4 Capturing numerical properties	28
Three remarks on notation · A remark about extensionality · The language L_A · A quick remark about truth · Expressing numerical properties and relations · Capturing numerical properties and relations · Expressing vs. capturing: keeping the distinction clear	
5 The truths of arithmetic	37
Sufficiently expressive languages · More about effectively enumerable sets · The truths of arithmetic are not effectively enumerable · Incompleteness	
6 Sufficiently strong arithmetics	43
The idea of a 'sufficiently strong' theory · An undecidability theorem · Another incompleteness theorem	
7 Interlude: Taking stock	47
Comparing incompleteness arguments · A road-map	
8 Two formalized arithmetics	51
BA, Baby Arithmetic · BA is negation complete · Q, Robinson Arithmetic · Q is not complete · Why Q is interesting	
	vii

Contents

9	What Q can prove	58
	Systems of logic · Capturing <i>less-than-or-equal-to</i> in Q · Adding ' $\leq$ ' to Q · Q is order-adequate · Defining the Δ_0 , Σ_1 and Π_1 wffs · Some easy results · Q is Σ_1 -complete · Intriguing corollaries · Proving Q is order-adequate	
10	First-order Peano Arithmetic	71
	Induction and the Induction Schema · Induction and relations · Arguing using induction · Being more generous with induction · Summary overview of PA · Hoping for completeness? · Where we've got to · Is PA consistent?	
11	Primitive recursive functions	83
	Introducing the primitive recursive functions · Defining the p.r. functions more carefully · An aside about extensionality · The p.r. functions are computable · Not all computable numerical functions are p.r. · Defining p.r. properties and relations · Building more p.r. functions and relations · Further examples	
12	Capturing p.r. functions	99
	Capturing a function · Two more ways of capturing a function · Relating our definitions · The idea of p.r. adequacy	
13	Q is p.r. adequate	106
	More definitions · Q can capture all Σ_1 functions · L_A can express all p.r. functions: starting the proof · The idea of a β -function · L_A can express all p.r. functions: finishing the proof · The p.r. functions are Σ_1 · The adequacy theorem · Canonically capturing	
14	Interlude: A very little about <i>Principia</i>	118
	<i>Principia</i> 's logicism · Gödel's impact · Another road-map	
15	The arithmetization of syntax	124
	Gödel numbering · Coding sequences · <i>Term</i> , <i>Atom</i> , <i>Wff</i> , <i>Sent</i> and <i>Prf</i> are p.r. · Some cute notation · The idea of diagonalization · The concatenation function · Proving that <i>Term</i> is p.r. · Proving that <i>Atom</i> and <i>Wff</i> are p.r. · Proving <i>Prf</i> is p.r.	
16	PA is incomplete	138
	Reminders · ' G is true if and only if it is unprovable' · PA is incomplete: the semantic argument · ' G is of Goldbach type' · Starting the syntactic argument for incompleteness · ω -incompleteness, ω -inconsistency · Finishing the syntactic argument · 'Gödel sentences' and what they say	
17	Gödel's First Theorem	147

Contents

Generalizing the semantic argument · Incompleteness · Generalizing the syntactic argument · The First Theorem	
18 Interlude: About the First Theorem	153
What we've proved · The reach of Gödelian incompleteness · Some ways to argue that G_T is true · What doesn't follow from incompleteness · What does follow from incompleteness?	
19 Strengthening the First Theorem	162
Broadening the scope of the incompleteness theorems · True Basic Arithmetic can't be axiomatized · Rosser's improvement · 1-consistency and Σ_1 -soundness	
20 The Diagonalization Lemma	169
Provability predicates · An easy theorem about provability predicates · G and Prov · Proving that G is equivalent to $\neg\text{Prov}(\ulcorner G \urcorner)$ · Deriving the Lemma	
21 Using the Diagonalization Lemma	175
The First Theorem again · An aside: 'Gödel sentences' again · The Gödel-Rosser Theorem again · Capturing provability? · Tarski's Theorem · The Master Argument · The length of proofs	
22 Second-order arithmetics	186
Second-order arithmetical languages · The Induction Axiom · Neat arithmetics · Introducing PA_2 · Categoricity · Incompleteness and categoricity · Another arithmetic · Speed-up again	
23 Interlude: Incompleteness and Isaacson's conjecture	199
Taking stock · Goodstein's Theorem · Isaacson's conjecture · Ever upwards · Ancestral arithmetic	
24 Gödel's Second Theorem for PA	212
Defining Con · The Formalized First Theorem in PA · The Second Theorem for PA · On ω -incompleteness and ω -consistency again · How should we interpret the Second Theorem? · How interesting is the Second Theorem for PA? · Proving the consistency of PA	
25 The derivability conditions	222
More notation · The Hilbert-Bernays-Löb derivability conditions · G , Con , and 'Gödel sentences' · Incompleteness and consistency extensions · The equivalence of fixed points for $\neg\text{Prov}$ · Theories that 'prove' their own inconsistency · Löb's Theorem	

Contents

26	Deriving the derivability conditions	232
	Nice* theories · The second derivability condition · The third derivability condition · Useful corollaries · The Second Theorem for weaker arithmetics · Jeroslow's Lemma and the Second Theorem	
27	Reflections	240
	The Second Theorem: the story so far · There are provable consistency sentences · What does that show? · The reflection schema: some definitions · Reflection and PA · Reflection, more generally · 'The best and most general version' · Another route to accepting a Gödel sentence?	
28	Interlude: About the Second Theorem	252
	'Real' vs 'ideal' mathematics · A quick aside: Gödel's caution · Relating the real and the ideal · Proving real-soundness? · The impact of Gödel · Minds and computers · The rest of this book: another road-map	
29	μ -Recursive functions	265
	Minimization and μ -recursive functions · Another definition of μ -recursiveness · The Ackermann-Péter function · The Ackermann-Péter function is μ -recursive · Introducing Church's Thesis · Why can't we diagonalize out? · Using Church's Thesis	
30	Undecidability and incompleteness	277
	Q is recursively adequate · Nice theories can <i>only</i> capture recursive functions · Some more definitions · Q and PA are undecidable · The <i>Entscheidungsproblem</i> · Incompleteness theorems again · Negation-complete theories are recursively decidable · Recursively adequate theories are not recursively decidable · What's next?	
31	Turing machines	287
	The basic conception · Turing computation defined more carefully · Some simple examples · 'Turing machines' and their 'states'	
32	Turing machines and recursiveness	298
	μ -Recursiveness entails Turing computability · μ -Recursiveness entails Turing computability: the details · Turing computability entails μ -recursiveness · Generalizing	
33	Halting problems	305
	Two simple results about Turing programs · The halting problem · The <i>Entscheidungsproblem</i> again · The halting problem and incompleteness · Another incompleteness argument · Kleene's Normal Form Theorem · Kleene's Theorem entails Gödel's First Theorem	

Contents

34	The Church–Turing Thesis	315
	From Euclid to Hilbert · 1936 and all that · What the Church–Turing Thesis is not · The status of the Thesis	
35	Proving the Thesis?	324
	The project · Vagueness and the idea of computability · Formal proofs and informal demonstrations · Squeezing arguments · The first premiss for a squeezing argument · The other premisses, thanks to Kolmogorov and Uspenskii · The squeezing argument defended · To summarize	
36	Looking back	342
	Further reading	344
	Bibliography	346
	Index	356

Preface

In 1931, the young Kurt Gödel published his First and Second Incompleteness Theorems; very often, these are simply referred to as ‘Gödel’s Theorems’. His startling results settled (or at least, seemed to settle) some of the crucial questions of the day concerning the foundations of mathematics. They remain of the greatest significance for the philosophy of mathematics – though just what that significance is continues to be debated. It has also frequently been claimed that Gödel’s Theorems have a much wider impact on very general issues about language, truth and the mind.

This book gives proofs of the Theorems and related formal results, and touches – necessarily briefly – on some of their implications. Who is this book for? Roughly speaking, for those who want a lot more fine detail than you get in books for a general audience (the best of those is Franzén, 2005), but who find the rather forbidding presentations in classic texts in mathematical logic (like Mendelson, 1997) too short on explanatory scene-setting. So I hope philosophy students taking an advanced logic course will find the book useful, as will mathematicians who want a more accessible exposition.

But don’t be misled by the relatively relaxed style; don’t try to browse through too quickly. We do cover a lot of ground in quite a bit of detail, and new ideas often come thick and fast. Take things slowly!

Theorems are numbered in the standard way, so Theorem 16.2 is the second theorem in Chapter 16. The distinction between a ‘proof’, a ‘proof sketch’ and (occasionally) a ‘sketch of a proof sketch’ is blurry. The end of a proof or proof sketch is marked by $\square$.

I assume only a modest amount of background in logic. So we can’t cover, for example, material that presupposes a serious knowledge of model theory; which means that we don’t discuss (say) model-theoretic arguments for incompleteness. That’s a pity. But there’s a sequel planned for enthusiasts who want to know about such matters.

I originally intended to write a rather shorter book, leaving many of the formal details to be filled in from elsewhere. But while that plan might have suited some readers, I soon realized that it would seriously irritate others to be sent hither and thither to consult a variety of textbooks with different terminologies and different notations. So in the end, I have given more or less full proofs of most of the key results we cover. However, my original plan shows through in two ways. First, some proofs are still only partially sketched in. Second, I try to signal very clearly when the detailed proofs I do give can be skipped without much loss of

Preface

understanding. With judicious skimming, you should be able to follow the main formal themes of the book even if you start from a very modest background in logic. For those who want to fill in more details and test their understanding there are exercises on the book's website at www.godelbook.net.

As we go through, there is also a small amount of broadly philosophical commentary. I follow Gödel in believing that our formal investigations and our general reflections on foundational matters should illuminate and guide each other. I hope that the more philosophical discussions – relatively elementary though certainly not always uncontentious – will also be reasonably widely accessible. Note however that I am more interested in patterns of ideas and arguments than in being historically very precise when talking e.g. about logicism or about Hilbert's Programme.

Writing a book like this presents many problems of organization. At various points we will need to call upon some background ideas from general logical theory. Do we explain them all at once, up front? Or do we introduce them as we go along, when needed? Similarly we will also need to call upon some ideas from the general theory of computation – for example, we will make use of both the notion of a 'primitive recursive function' and the more general notion of a ' μ -recursive function'. Again, do we explain these together? Or do we give the explanations many chapters apart, when the respective notions first get used?

I've mostly adopted the second policy, introducing new ideas as and when needed. This has its costs, but I think that there is a major compensating benefit, namely that the way the book is organized makes it clearer just what depends on what. It also reflects something of the historical order in which ideas emerged.

My colleague Michael Potter has been an inspiring presence since I returned to Cambridge. Many thanks are due to him and to all those who have very kindly given me comments on parts of various drafts, including the late Torkel Franzén, Tim Button, Luca Incurvati, Jeffrey Ketland, Aatu Koskensilta, Christopher Leary, Mary Leng, Toby Ord, Alex Paseau, Jacob Plotkin, José F. Ruiz, Kevin Scharp, Hartley Slater, and Tim Storer. I should especially mention Richard Zach, whose comments at two different stages were particularly extensive and particularly helpful.

But my greatest debt is to Patsy Wilson-Smith, without whose continuing love and support this book would never have been written.